Taylorův polynom

Taylorův rozvoj v bodě $\mathcal{A} = [x_0, \dots, z_0]$ pro funkci více (reálných) proměnných (která má spojité parciální derivace) vyjádříme pomocí totálních diferenciálů jako

$$\underline{T_n(x, \dots, z) = f(x_0, \dots, z_0) + \frac{\mathrm{d}f(x_0, \dots, z_0)}{1!} + \frac{\mathrm{d}^2 f(x_0, \dots, z_0)}{2!} + \frac{\mathrm{d}^3 f(x_0, \dots, z_0)}{3!} + \cdots}$$

Taylorova řada se používá k polynomiální aproximaci funkcí, protože platí, že všechny derivace Taylorova polynomu stupně n mají ve středu rozvoje (v bodě $\mathcal{A}[x_0, \dots, z_0]$) stejné funkční hodnoty jako odpovídající derivace funkce f. Tato aproximace je na okolí bodu $\mathcal{A}$ (středu rozvoje) tím přesnější, čím vyšší stupeň polynomu použijeme. Zároveň platí, že se chyba se vzdáleností od středu zvyšuje.

V případě, že má Taylorův polynom střed v počátku, pak se označuje jako ***Maclaurinův*** polynom.

1. $[x \in \mathbb{R};\ y \in \mathbb{R}]$ V okolí středu $A = [1; -2]$ rozviňte do **Taylorova mnohočlenu** (nultého, prvého, druhého, třetího a čtvrtého stupně) funkci:

$$f(x, y) = 6xy^2 - 2x^3 - 3y^3 .$$

$f(1; -2) = 6 \cdot 1 \cdot (-2)^2 - 2 \cdot 1^3 - 3 \cdot (-2)^3 = 24 - 2 + 24 = 46$

$$\underline{\underline{T_0(A) = 46}}$$

$f_x(1; -2) = 18 \quad f_x = 6y^2 - 6x^2$
$f_y(1; -2) = -60 \quad f_y = 12xy - 9y^2$

$\mathrm{d}f(1; -2) = 18 \cdot (x - 1) - 60 \cdot (y + 2)$

$$\underline{\underline{T_1(A) = 46 + 18 \cdot (x - 1) - 60 \cdot (y + 2)}}$$

$f_{xx}(1; -2) = -12 \quad f_{xx} = -12x$
$f_{xy}(1; -2) = -24 \quad f_{xy} = 12y$
$f_{yx}(1; -2) = -24 \quad f_{yx} = 12y$
$f_{yy}(1; -2) = 48 \quad f_{yy} = 12x - 18y$

$$\mathrm{d}^2 f(1; -2) = -12 \cdot (x - 1) \cdot (x - 1) - 24 \cdot (x - 1) \cdot (y + 2) - 24 \cdot (y + 2) \cdot (x - 1) + \\ + 48 \cdot (y + 2) \cdot (y + 2) = -12 \cdot (x - 1)^2 - 48 \cdot (x - 1)(y + 2) + 48 \cdot (y + 2)^2$$

$$\underline{\underline{T_2(A) = 46 + 18.(x - 1) - 60.(y + 2) - 6.(x - 1)^2 - 24.(x - 1).(y + 2) + 24.(y + 2)^2}}$$

$$
\begin{array}{ll}
f_{xxx}(1;-2) = -12 & f_{xxx} = -12 \\
f_{xxy}(1;-2) = 0 & f_{xxy} = 0 \\
\hline
f_{xyx}(1;-2) = 0 & f_{xyx} = 0 \\
f_{xyy}(1;-2) = 12 & f_{xyy} = 12 \\
\hline
f_{yxx}(1;-2) = 0 & f_{yxx} = 0 \\
f_{yxy}(1;-2) = 12 & f_{yxy} = 12 \\
\hline
f_{yyx}(1;-2) = 12 & f_{yyx} = 12 \\
f_{yyy}(1;-2) = -18 & f_{yyy} = -18
\end{array}
$$

$$
\begin{aligned}
\mathrm{d}^3 f(1;-2) &= -12 \cdot (x-1) \cdot (x-1) \cdot (x-1) + 0 + 12 \cdot (x-1) \cdot (y+2) \cdot (y+2) + \\
&+ 12 \cdot (y+2) \cdot (x-1) \cdot (y+2) + 12 \cdot (y+2) \cdot (y+2) \cdot (x-1) - 18 \cdot (y+2) \cdot (y+2) \cdot (y+2) = \\
&= -12 \cdot (x-1)^3 + 36 \cdot (x-1) \cdot (y+2)^2 - 18 \cdot (y+2)^3
\end{aligned}
$$

$$
\begin{aligned}
T_3(A) &= 46 + 18 \cdot (x-1) - 60 \cdot (y+2) - 6 \cdot (x-1)^2 - 24 \cdot (x-1) \cdot (y+2) + \\
&+ 24 \cdot (y+2)^2 - 2 \cdot (x-1)^3 + 6 \cdot (x-1) \cdot (y+2)^2 - 3 \cdot (y+2)^3
\end{aligned}
$$

$$
\begin{aligned}
f_{xxxx} &= f_{xxxy} = 0 \\
f_{xxyx} &= f_{xxyy} = 0 \\
f_{xyxx} &= f_{xyxy} = 0 \\
f_{xyyx} &= f_{xyyy} = 0 \\
f_{yxxx} &= f_{yxxy} = 0 \\
f_{yxyx} &= f_{yxyy} = 0 \\
f_{yyxx} &= f_{yyxy} = 0 \\
f_{yyyx} &= f_{yyyy} = 0
\end{aligned}
$$

$$\mathrm{d}^4 f(1;-2) = 0$$

$$
\begin{aligned}
T_4(A) &= 46 + 18 \cdot (x-1) - 60 \cdot (y+2) - 6 \cdot (x-1)^2 - 24 \cdot (x-1) \cdot (y+2) + \\
&+ 24 \cdot (y+2)^2 - 2 \cdot (x-1)^3 + 6 \cdot (x-1) \cdot (y+2)^2 - 3 \cdot (y+2)^3 + 0 \quad = T_3(A)
\end{aligned}
$$

$$\mathrm{d}^5 f(1;-2) = 0$$

$$\vdots$$

$$T_3(A) = T_4(A) = T_5(A) = T_6(A) = \dots$$

2. a) $[x \in \mathbb{R};\ y \in \mathbb{R}]$ V okolí středu $B = [5;6]$ rozviňte do **Taylorovy** řady funkci

$$f(x,y) = x^3 + y^2 - 6xy - 39x + 18y + 4$$

$$f(5;6) = 5^3 + 6^2 - 6\cdot 5\cdot 6 - 39\cdot 5 + 18\cdot 6 + 4 = 125 + 36 - 180 - 195 + 108 + 4 = -102$$

$$\begin{array}{ll} f_x(5;6) = 0 & f_x = 3x^2 - 6y - 39 \\ f_y(5;6) = 0 & f_y = 2y - 6x + 18 \end{array}$$

$$\mathrm{d}f(5;6) = 0\cdot(x-1) + 0\cdot(y+2) = 0$$

$$\begin{array}{ll} f_{xx}(5;6) = 30 & f_{xx} = 6x \\ f_{xy}(5;6) = -6 & f_{xy} = -6 \\ \hline f_{yx}(5;6) = -6 & f_{yx} = -6 \\ f_{yy}(5;6) = 2 & f_{yy} = 2 \end{array}$$

$$\mathrm{d}^2f(5;6) = 30\cdot(x-5)\cdot(x-5) - 6\cdot(x-5)\cdot(y-6) - 6\cdot(y-6)\cdot(x-5) +$$
$$+ 2\cdot(y-6)\cdot(y-6) = 30(x-5)^2 - 12(x-5)(y-6) + 2(y-6)^2$$

$$\begin{array}{ll} f_{xxx}(5;6) = 6 & f_{xxx} = 6 \\ f_{xxy}(5;6) = 0 & f_{xxy} = 0 \\ \hline f_{xyx}(5;6) = 0 & f_{xyx} = 0 \\ f_{xyy}(5;6) = 0 & f_{xyy} = 0 \\ \hline f_{yxx}(5;6) = 0 & f_{yxx} = 0 \\ f_{yxy}(5;6) = 0 & f_{yxy} = 0 \\ \hline f_{yyx}(5;6) = 0 & f_{yyx} = 0 \\ f_{yyy}(5;6) = 0 & f_{yyy} = 0 \end{array}$$

$$\mathrm{d}^3f(5;6) = 6\cdot(x-5)\cdot(x-5)\cdot(x-5) + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 6(x-5)^3$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4f(5;6) = 0;\ \mathrm{d}^5f(5;6) = 0;\ \mathrm{d}^6f(5;6) = 0;\ \mathrm{d}^7f(5;6) = 0;\ \ldots \quad \text{proto}$$

$$\underline{\underline{T(B) =}}\ f(5;6) + \frac{\mathrm{d}f(5;6)}{1!} + \frac{\mathrm{d}^2f(5;6)}{2!} + \frac{\mathrm{d}^3f(5;6)}{3!} =$$
$$\underline{\underline{= -102 + 15(x-5)^2 - 6(x-5)(y-6) + (y-6)^2 + (x-5)^3}}$$

2. b) $[x \in \mathbb{R};\ y \in \mathbb{R}]$ V okolí středu $C = [1;-2]$ rozviňte do **Taylorovy** řady funkci

$$f(x,y) = x^3 + y^2 - 6xy - 39x + 18y + 4$$

$$f(1;-2) = 1^3+(-2)^2-6\cdot1\cdot(-2)-39\cdot1+18\cdot(-2)+4 = 1+4+12-39-36+4 = -54$$

$f_x(1;-2) = -24$ $\quad f_x = 3x^2 - 6y - 39$
$f_y(1;-2) = 8$ $\quad f_y = 2y - 6x + 18$

$$\mathrm{d}f(1;-2) = -24\cdot(x-1) + 8\cdot(y+2)$$

$f_{xx}(1;-2) = 6$ $\quad f_{xx} = 6x$
$f_{xy}(1;-2) = -6$ $\quad f_{xy} = -6$
$f_{yx}(1;-2) = -6$ $\quad f_{yx} = -6$
$f_{yy}(1;-2) = 2$ $\quad f_{yy} = 2$

$$\mathrm{d}^2f(1;-2) = 6\cdot(x-1)\cdot(x-1) - 6\cdot(x-1)\cdot(y+2) - 6\cdot(y+2)\cdot(x-1) + \\ + 2\cdot(y+2)\cdot(y+2) = 6(x-1)^2 - 12(x-5)(y+2) + 2(y+2)^2$$

$f_{xxx}(1;-2) = 6$ $\quad f_{xxx} = 6$
$f_{xxy}(1;-2) = 0$ $\quad f_{xxy} = 0$
$f_{xyx}(1;-2) = 0$ $\quad f_{xyx} = 0$
$f_{xyy}(1;-2) = 0$ $\quad f_{xyy} = 0$
$f_{yxx}(1;-2) = 0$ $\quad f_{yxx} = 0$
$f_{yxy}(1;-2) = 0$ $\quad f_{yxy} = 0$
$f_{yyx}(1;-2) = 0$ $\quad f_{yyx} = 0$
$f_{yyy}(1;-2) = 0$ $\quad f_{yyy} = 0$

$$\mathrm{d}^3f(1;-2) = 6\cdot(x-1)\cdot(x-1)\cdot(x-1) + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 6(x-1)^3$$

Všechny další parciální derivace jsou rovny nule,

$\mathrm{d}^4f(1;-2) = 0;\ \mathrm{d}^5f(1;-2) = 0;\ \mathrm{d}^6f(1;-2) = 0;\ \mathrm{d}^7f(1;-2) = 0;\ \ldots$ proto

$$\underline{\underline{T(C) =}} \quad f(1;-2) + \frac{\mathrm{d}f(1;-2)}{1!} + \frac{\mathrm{d}^2f(1;-2)}{2!} + \frac{\mathrm{d}^3f(1;-2)}{3!} =$$

$$\underline{\underline{= -54 + -24(x-1) + 8(y+2) + 3(x-1)^2 - 6(x-1)(y+2) + (y+2)^2 + (x-1)^3}}$$

2. c) $[x \in \mathbb{R};\ y \in \mathbb{R}]$ V okolí středu $D = [1;2]$ rozviňte do **Taylorovy** řady funkci

$$f(x,y) = x^3 + y^2 - 6xy - 39x + 18y + 4$$

$$f(1;-2) = 1^3 + 2^2 - 6 \cdot 1 \cdot 2 - 39 \cdot 1 + 18 \cdot 2 + 4 = 1 + 4 - 12 - 39 + 36 + 4 = -6$$

$f_x(1;2) = -48 \quad f_x = 3x^2 - 6y - 39$

$f_y(1;2) = 16 \quad f_y = 2y - 6x + 18$

$$\mathrm{d}f(1;2) = -48 \cdot (x-1) + 16 \cdot (y-2)$$

$f_{xx}(1;2) = 6 \quad f_{xx} = 6x$

$f_{xy}(1;2) = -6 \quad f_{xy} = -6$

$f_{yx}(1;2) = -6 \quad f_{yx} = -6$

$f_{yy}(1;2) = 2 \quad f_{yy} = 2$

$$\begin{aligned}\mathrm{d}^2f(1;2) &= 6 \cdot (x-1) \cdot (x-1) - 6 \cdot (x-1) \cdot (y-2) - 6 \cdot (y-2) \cdot (x-1) + \\ &\quad + 2 \cdot (y-2) \cdot (y+2) = 6(x-1)^2 - 12(x-5)(y-2) + 2(y-2)^2\end{aligned}$$

$f_{xxx}(1;2) = 6 \quad f_{xxx} = 6$

$f_{xxy}(1;2) = 0 \quad f_{xxy} = 0$

$f_{xyx}(1;2) = 0 \quad f_{xyx} = 0$

$f_{xyy}(1;2) = 0 \quad f_{xyy} = 0$

$f_{yxx}(1;2) = 0 \quad f_{yxx} = 0$

$f_{yxy}(1;2) = 0 \quad f_{yxy} = 0$

$f_{yyx}(1;2) = 0 \quad f_{yyx} = 0$

$f_{yyy}(1;2) = 0 \quad f_{yyy} = 0$

$$\mathrm{d}^3f(1;2) = 6 \cdot (x-1) \cdot (x-1) \cdot (x-1) + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 6(x-1)^3$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4f(1;2) = 0;\ \mathrm{d}^5f(1;2) = 0;\ \mathrm{d}^6f(1;2) = 0;\ \mathrm{d}^7f(1;2) = 0;\ \dots \quad \text{proto}$$

$$\underline{\underline{T(D) =}} \quad f(1;2) + \frac{\mathrm{d}f(1;2)}{1!} + \frac{\mathrm{d}^2f(1;2)}{2!} + \frac{\mathrm{d}^3f(1;2)}{3!} =$$

$$\underline{\underline{= -6 - 48(x-1) + 16(y-2) + 3(x-1)^2 - 6(x-1)(y-2) + (y-2)^2 + (x-1)^3}}$$

3. $[x \in \mathbb{R};\ y \in \mathbb{R}]$ Rozviňte do **Maclaurinovy** řady ($\Rightarrow$ Taylorův rozvoj se středem v počátku $O = [0; 0]$) funkci $f(x, y) = \mathrm{e}^{x+y}$ (viz poznámka).

$f(0; 0) = \mathrm{e}^{0+0} = \mathrm{e}^0 = 1$

$f_x(0; 0) = 1 \quad f_x = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$
$f_y(0; 0) = 1 \quad f_y = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$

$\mathrm{d}f(0; 0) = 1 \cdot (x - 0) + 1 \cdot (y - 0) = x + y$

$f_{xx}(0; 0) = 1 \quad f_{xx} = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$
$f_{xy}(0; 0) = 1 \quad f_{xy} = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$
$f_{yx}(0; 0) = 1 \quad f_{yx} = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$
$f_{yy}(0; 0) = 1 \quad f_{yy} = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$

$\mathrm{d}^2 f(0; 0) = 1 \cdot x \cdot x + 1 \cdot x \cdot y + 1 \cdot y \cdot x + 1 \cdot y \cdot y = x^2 + 2xy + y^2 = (x + y)^2$

$f_{xxx}(0; 0) = 1 \quad f_{xxx} = \mathrm{e}^{x+y}$
$f_{xxy}(0; 0) = 1 \quad f_{xxy} = \mathrm{e}^{x+y}$
$f_{xyx}(0; 0) = 1 \quad f_{xyx} = \mathrm{e}^{x+y}$
$f_{xyy}(0; 0) = 1 \quad f_{xyy} = \mathrm{e}^{x+y}$
$f_{yxx}(0; 0) = 1 \quad f_{yxx} = \mathrm{e}^{x+y}$
$f_{yxy}(0; 0) = 1 \quad f_{yxy} = \mathrm{e}^{x+y}$
$f_{yyx}(0; 0) = 1 \quad f_{yyx} = \mathrm{e}^{x+y}$
$f_{yyy}(0; 0) = 1 \quad f_{yyy} = \mathrm{e}^{x+y}$

$$\mathrm{d}^3 f(0; 0) = 1 \cdot x \cdot x \cdot x + 1 \cdot x \cdot x \cdot y + 1 \cdot x \cdot y \cdot x + 1 \cdot x \cdot y \cdot y + 1 \cdot y \cdot x \cdot x + 1 \cdot y \cdot x \cdot y + 1 \cdot y \cdot y \cdot x + 1 \cdot y \cdot y \cdot y = x^3 + 3x^2 y + 3xy^2 + y^3 = (x + y)^3$$

$f_{xxxx}(0; 0) = 1 \quad f_{xxxx} = \mathrm{e}^{x+y}$
$f_{xxxy}(0; 0) = 1 \quad f_{xxxy} = \mathrm{e}^{x+y}$
$f_{xxyx}(0; 0) = 1 \quad f_{xxyx} = \mathrm{e}^{x+y}$
$f_{xxyy}(0; 0) = 1 \quad f_{xxyy} = \mathrm{e}^{x+y}$
$f_{xyxx}(0; 0) = 1 \quad f_{xyxx} = \mathrm{e}^{x+y}$
$f_{xyxy}(0; 0) = 1 \quad f_{xyxy} = \mathrm{e}^{x+y}$
$f_{xyyx}(0; 0) = 1 \quad f_{xyyx} = \mathrm{e}^{x+y}$
$f_{xyyy}(0; 0) = 1 \quad f_{xyyy} = \mathrm{e}^{x+y}$
$f_{yxxx}(0; 0) = 1 \quad f_{yxxx} = \mathrm{e}^{x+y}$
$f_{yxxy}(0; 0) = 1 \quad f_{yxxy} = \mathrm{e}^{x+y}$
$f_{yxyx}(0; 0) = 1 \quad f_{yxyx} = \mathrm{e}^{x+y}$
$f_{yxyy}(0; 0) = 1 \quad f_{yxyy} = \mathrm{e}^{x+y}$
$f_{yyxx}(0; 0) = 1 \quad f_{yyxx} = \mathrm{e}^{x+y}$
$f_{yyxy}(0; 0) = 1 \quad f_{yyxy} = \mathrm{e}^{x+y}$
$f_{yyyx}(0; 0) = 1 \quad f_{yyyx} = \mathrm{e}^{x+y}$
$f_{yyyy}(0; 0) = 1 \quad f_{yyyy} = \mathrm{e}^{x+y}$

$$\mathrm{d}^4 f(0; 0) = 1 \cdot x \cdot x \cdot x \cdot x + 1 \cdot x \cdot x \cdot x \cdot y + 1 \cdot x \cdot x \cdot y \cdot x + 1 \cdot x \cdot x \cdot y \cdot y + 1 \cdot x \cdot y \cdot x \cdot x + 1 \cdot x \cdot y \cdot x \cdot y + 1 \cdot x \cdot y \cdot y \cdot x + 1 \cdot x \cdot y \cdot y \cdot y + 1 \cdot y \cdot x \cdot x \cdot x + 1 \cdot y \cdot x \cdot x \cdot y + 1 \cdot y \cdot x \cdot y \cdot x + 1 \cdot y \cdot x \cdot y \cdot y + 1 \cdot y \cdot y \cdot x \cdot x + 1 \cdot y \cdot y \cdot x \cdot y + 1 \cdot y \cdot y \cdot y \cdot x + 1 \cdot y \cdot y \cdot y \cdot y = x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4 = (x + y)^4$$

$$M = T(O) = 1 + \frac{x + y}{1!} + \frac{(x + y)^2}{2!} + \frac{(x + y)^3}{3!} + \frac{(x + y)^4}{4!} + \frac{(x + y)^5}{5!} + \frac{(x + y)^6}{6!} + \cdots$$

4. $[x \in \mathbb{R};\ y \in \mathbb{R}]$ V okolí středu $B = [2;1]$ rozviňte do **Taylorovy** řady funkci

$$f(x,y) = y^3 + 3x^2 - 3y^2 - 3x^2y + 12xy - 11x - 9y + 9$$

$$f(2;1) = 1^3 + 3\cdot 2^2 - 3\cdot 1^2 - 3\cdot 2^2\cdot 1 + 12\cdot 2\cdot 1 - 11\cdot 2 - 9\cdot 1 + 9 =$$
$$= 1 + 12 - 3 - 12 + 24 - 22 - 9 + 9 = 0$$

$f_x(2;1) = 1 \quad f_x = 6x - 6xy + 12y - 11$
$f_y(2;1) = 0 \quad f_y = 3y^2 - 6y - 3x^2 + 12x - 9$

$$\mathrm{d}f(2;1) = 1\cdot(x-2) + 0\cdot(y-1) = (x-2)$$

$f_{xx}(2;1) = 0 \quad f_{xx} = 6 - 6y$
$f_{xy}(2;1) = 0 \quad f_{xy} = -6x + 12$
$f_{yx}(2;1) = 0 \quad f_{yx} = -6x + 12$
$f_{yy}(2;1) = 0 \quad f_{yy} = 6y - 6$

$$\mathrm{d}^2f(1;2) = 0\cdot(x-2)\cdot(x-2) + 0\cdot(x-2)\cdot(y-1) + 0\cdot(y-1)\cdot(x-2) +$$
$$+ 0\cdot(y-1)\cdot(y-1) = 0$$

$f_{xxx}(2;1) = 0 \quad f_{xxx} = 0$
$f_{xxy}(2;1) = -6 \quad f_{xxy} = -6$
$f_{xyx}(2;1) = -6 \quad f_{xyx} = -6$
$f_{xyy}(2;1) = 0 \quad f_{xyy} = 0$
$f_{yxx}(2;1) = -6 \quad f_{yxx} = -6$
$f_{yxy}(2;1) = 0 \quad f_{yxy} = 0$
$f_{yyx}(2;1) = 0 \quad f_{yyx} = 0$
$f_{yyy}(2;1) = 6 \quad f_{yyy} = 6$

$$\mathrm{d}^3f(2;1) = 0 - 6\cdot(x-2)\cdot(x-2)\cdot(y-1) - 6\cdot(x-2)\cdot(y-1)\cdot(x-2) + 0 -$$
$$- 6\cdot(y-1)\cdot(x-2)\cdot(x-2) + 0 + 0 + 6\cdot(y-1)\cdot(y-1)\cdot(y-1) =$$
$$= 6(x-2)^3 - 18(x-2)(y-1)^2$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4f(2;1) = 0;\ \mathrm{d}^5f(2;1) = 0;\ \mathrm{d}^6f(2;1) = 0;\ \mathrm{d}^7f(2;1) = 0;\ \dots \quad \text{proto}$$

$$\underline{\underline{T(B) =}} \quad f(2;1) + \frac{\mathrm{d}f(2;1)}{1!} + \frac{\mathrm{d}^2f(2;1)}{2!} + \frac{\mathrm{d}^3f(2;1)}{3!} =$$
$$\underline{\underline{(x-2) - 3.(y-1).(x-2)^2 + (y-1)^3}}$$

5. $[x \in \mathbb{R};\ y \in \mathbb{R}]$ Rozviňte do **Maclaurinovy** řady funkci $f(x,y) = \sin(x+y)$ (viz poznámka).

$f(0;0) = \sin(0+0) = \sin 0 = 0$

$f_x(0;0) = 1 \quad f_x = \cos(x+y)\cdot 1 = \cos(x+y)$
$f_y(0;0) = 1 \quad f_y = \cos(x+y)\cdot 1 = \cos(x+y)$

$\mathrm{d}f(0;0) = 1\cdot(x-0) + 1\cdot(y-0) = x+y$

$f_{xx}(0;0) = 0 \quad f_{xx} = -\sin(x+y)\cdot 1 = -\sin(x+y)$
$f_{xy}(0;0) = 0 \quad f_{xy} = -\sin(x+y)\cdot 1 = -\sin(x+y)$
$f_{yx}(0;0) = 0 \quad f_{yx} = -\sin(x+y)\cdot 1 = -\sin(x+y)$
$f_{yy}(0;0) = 0 \quad f_{yy} = -\sin(x+y)\cdot 1 = -\sin(x+y)$

$\mathrm{d}^2 f(0;0) = 0\cdot x\cdot x + 0\cdot x\cdot y + 0\cdot y\cdot x + 0\cdot y\cdot y = 0$

$f_{xxx}(0;0) = -1 \quad f_{xxx} = -\cos(x+y)$
$f_{xxy}(0;0) = -1 \quad f_{xxy} = -\cos(x+y)$
$f_{xyx}(0;0) = -1 \quad f_{xyx} = -\cos(x+y)$
$f_{xyy}(0;0) = -1 \quad f_{xyy} = -\cos(x+y)$
$f_{yxx}(0;0) = -1 \quad f_{yxx} = -\cos(x+y)$
$f_{yxy}(0;0) = -1 \quad f_{yxy} = -\cos(x+y)$
$f_{yyx}(0;0) = -1 \quad f_{yyx} = -\cos(x+y)$
$f_{yyy}(0;0) = -1 \quad f_{yyy} = -\cos(x+y)$

$$\mathrm{d}^3 f(0;0) = -1\cdot x\cdot x\cdot x - 1\cdot x\cdot x\cdot y - 1\cdot x\cdot y\cdot x - 1\cdot x\cdot y\cdot y - 1\cdot y\cdot x\cdot x - 1\cdot y\cdot x\cdot y + \\ -1\cdot y\cdot y\cdot x - 1\cdot y\cdot y\cdot y = -x^3 - 3x^2y - 3xy^2 - y^3 = -(x+y)^3$$

$f_{xxxx}(0;0) = 0 \quad f_{xxxx} = -[-\sin(x+y)] = \sin(x+y)$
$f_{xxxy}(0;0) = 0 \quad f_{xxxy} = \sin(x+y)$
$f_{xxyx}(0;0) = 0 \quad f_{xxyx} = \sin(x+y)$
$f_{xxyy}(0;0) = 0 \quad f_{xxyy} = \sin(x+y)$
$f_{xyxx}(0;0) = 0 \quad f_{xyxx} = \sin(x+y)$
$f_{xyxy}(0;0) = 0 \quad f_{xyxy} = \sin(x+y)$
$f_{xyyx}(0;0) = 0 \quad f_{xyyx} = \sin(x+y)$
$f_{xyyy}(0;0) = 0 \quad f_{xyyy} = \sin(x+y)$
$f_{yxxx}(0;0) = 0 \quad f_{yxxx} = \sin(x+y)$
$f_{yxxy}(0;0) = 0 \quad f_{yxxy} = \sin(x+y)$
$f_{yxyx}(0;0) = 0 \quad f_{yxyx} = \sin(x+y)$
$f_{yxyy}(0;0) = 0 \quad f_{yxyy} = \sin(x+y)$
$f_{yyxx}(0;0) = 0 \quad f_{yyxx} = \sin(x+y)$
$f_{yyxy}(0;0) = 0 \quad f_{yyxy} = \sin(x+y)$
$f_{yyyx}(0;0) = 0 \quad f_{yyyx} = \sin(x+y)$
$f_{yyyy}(0;0) = 0 \quad f_{yyyy} = \sin(x+y) \qquad f_{yyyyy} = \cos(x+y) \quad f_{yyyyy}(0;0) = 1$

$$\mathrm{d}^4 f(0;0) = 0\cdot x\cdot x\cdot x\cdot x + 0\cdot x\cdot x\cdot x\cdot y + 0\cdot x\cdot x\cdot y\cdot x + 0\cdot x\cdot x\cdot y\cdot y + \\ + 0\cdot x\cdot y\cdot x\cdot x + 0\cdot x\cdot y\cdot x\cdot y + 0\cdot x\cdot y\cdot y\cdot x + 0\cdot x\cdot y\cdot y\cdot y + 0\cdot y\cdot x\cdot x\cdot x + \\ + 0\cdot y\cdot x\cdot x\cdot y + 0\cdot y\cdot x\cdot y\cdot x + 0\cdot y\cdot x\cdot y\cdot y + 0\cdot y\cdot y\cdot x\cdot x + 0\cdot y\cdot y\cdot x\cdot y + \\ + 0\cdot y\cdot y\cdot y\cdot x + 0\cdot y\cdot y\cdot y\cdot y = 0$$

$$M = \frac{x+y}{1!} - \frac{(x+y)^3}{3!} + \frac{(x+y)^5}{5!} - \frac{(x+y)^7}{7!} + \cdots$$

Funkce sin je LICHÁ.

6. $[x \in \mathbb{R};\ y \in \mathbb{R}]$ V okolí středu $A = [1; 2]$ rozviňte do **Taylorovy** řady funkci

$$f(x, y) = x^3 - 3x^2 + 3y^2 - 3xy^2 + 12xy - 9x - 11y + 9$$

$$f(1; 2) = 1^3 - 3 \cdot 1^2 + 3 \cdot 2^2 - 3 \cdot 1 \cdot 2^2 + 12 \cdot 1 \cdot 2 - 9 \cdot 1 - 11 \cdot 2 + 9 =$$
$$= 1 - 3 + 12 - 12 + 24 - 9 - 22 + 9 = 0$$

$f_x(1; 2) = 0 \quad f_x = 3x^2 - 6x - 3y^2 + 12y - 9$
$f_y(1; 2) = 1 \quad f_y = 6y - 6xy + 12x - 11$

$$\mathrm{d}f(1; 2) = 0 \cdot (x - 1) + 1 \cdot (y - 2) = (y - 2)$$

$f_{xx}(1; 2) = 0 \quad f_{xx} = 6x - 6$
$f_{xy}(1; 2) = 0 \quad f_{xy} = -6y + 12$
$f_{yx}(1; 2) = 0 \quad f_{yx} = -6y + 12$
$f_{yy}(1; 2) = 0 \quad f_{yy} = 6 - 6x$

$$\mathrm{d}^2 f(1; 2) = 0 \cdot (x - 1) \cdot (x - 1) + 0 \cdot (x - 1) \cdot (y - 2) + 0 \cdot (y - 2) \cdot (x - 1) +$$
$$+ 0 \cdot (y - 2) \cdot (y - 2) = 0$$

$f_{xxx}(1; 2) = 6 \quad f_{xxx} = 6$
$f_{xxy}(1; 2) = 0 \quad f_{xxy} = 0$
$f_{xyx}(1; 2) = 0 \quad f_{xyx} = 0$
$f_{xyy}(1; 2) = -6 \quad f_{xyy} = -6$
$f_{yxx}(1; 2) = 0 \quad f_{yxx} = 0$
$f_{yxy}(1; 2) = -6 \quad f_{yxy} = -6$
$f_{yyx}(1; 2) = -6 \quad f_{yyx} = -6$
$f_{yyy}(1; 2) = 0 \quad f_{yyy} = 0$

$$\mathrm{d}^3 f(1; 2) = 6 \cdot (x - 1) \cdot (x - 1) \cdot (x - 1) + 0 + 0 - 6 \cdot (x - 1) \cdot (y - 2) \cdot (y - 2) + 0 -$$
$$- 6 \cdot (y - 2) \cdot (x - 1) \cdot (y - 2) - 6 \cdot (y - 2) \cdot (y - 2) \cdot (x - 1) + 0 =$$
$$= 6(x - 1)^3 - 18(x - 1)(y - 2)^2$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^6 f(1; 2) = 0;\ \mathrm{d}^5 f(1; 2) = 0;\ \mathrm{d}^6 f(1; 2) = 0;\ \mathrm{d}^7 f(1; 2) = 0;\ \dots \quad \text{proto}$$

$$\underline{\underline{T(A) =}} \quad f(1; 2) + \frac{\mathrm{d}f(1; 2)}{1!} + \frac{\mathrm{d}^2 f(1; 2)}{2!} + \frac{\mathrm{d}^3 f(1; 2)}{3!} =$$
$$\underline{\underline{(y - 2) + (x - 1)^3 - 3.(x - 1).(y - 2)^2}}$$

7. $[x \in \mathbb{R};\ y \in \mathbb{R}]$ Rozviňte do **Maclaurinovy** řady funkci $f(x,y) = \mathrm{e}^{xy}$

(viz poznámka).

$f(0;0) = \mathrm{e}^{0\cdot 0} = \mathrm{e}^0 = 1$

$f_x(0;0) = 0 \quad f_x = \mathrm{e}^{xy} \cdot y = y \cdot \mathrm{e}^{xy}$
$f_y(0;0) = 0 \quad f_y = \mathrm{e}^{xy} \cdot x = x \cdot \mathrm{e}^{xy}$

$\mathrm{d}f(0;0) = 0 \cdot (x-0) + 0 \cdot (y-0) = 0$

$f_{xx}(0;0) = 0 \quad f_{xx} = y \cdot \mathrm{e}^{xy} \cdot y = y^2 \cdot \mathrm{e}^{xy}$
$f_{xy}(0;0) = 1 \quad f_{xy} = 1 \cdot \mathrm{e}^{xy} + y \cdot \mathrm{e}^{xy} \cdot x = \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy}$
$f_{yx}(0;0) = 1 \quad f_{yx} = 1 \cdot \mathrm{e}^{xy} + x \cdot \mathrm{e}^{xy} \cdot y = \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy}$
$f_{yy}(0;0) = 0 \quad f_{yy} = x \cdot \mathrm{e}^{xy} \cdot x = x^2 \cdot \mathrm{e}^{xy}$

Je lepší výsledky před dalším derivováním co nejvíce upravit tak, jako v následujících příkladech v tomto dokumentu, nebo ve 3. příkladu, 9. cvičení na parciální derivace.

$\mathrm{d}^2 f(0;0) = 0 \cdot x \cdot x + 1 \cdot x \cdot y + 1 \cdot y \cdot x + 0 \cdot y \cdot y = 2xy = 2! \cdot \frac{xy}{1!}$

$f_{xxx}(0;0) = 0 \quad f_{xxx} = y^2 \cdot \mathrm{e}^{xy} \cdot y = y^3 \cdot \mathrm{e}^{xy}$
$f_{xxy}(0;0) = 0 \quad f_{xxy} = 2y \cdot \mathrm{e}^{xy} + y^2 \cdot \mathrm{e}^{xy} \cdot x = 2y \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy}$
$f_{xyx}(0;0) = 0 \quad f_{xyx} = \mathrm{e}^{xy} \cdot y + y \cdot \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy} \cdot y = 2y \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy}$
$f_{xyy}(0;0) = 0 \quad f_{xyy} = \mathrm{e}^{xy} \cdot x + x \cdot \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy} \cdot x = 2x \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy}$
$f_{yxx}(0;0) = 0 \quad f_{yxx} = \mathrm{e}^{xy} \cdot y + y \cdot \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy} \cdot y = 2y \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy}$
$f_{yxy}(0;0) = 0 \quad f_{yxy} = \mathrm{e}^{xy} \cdot x + x \cdot \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy} \cdot x = 2x \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy}$
$f_{yyx}(0;0) = 0 \quad f_{yyx} = 2x \cdot \mathrm{e}^{xy} + x^2 \cdot \mathrm{e}^{xy} \cdot y = 2x \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy}$
$f_{yyy}(0;0) = 0 \quad f_{yyy} = x^2 \cdot \mathrm{e}^{xy} \cdot x = x^3 \cdot \mathrm{e}^{xy}$

$\mathrm{d}^3 f(0;0) = 0 \cdot x \cdot x \cdot x + 0 \cdot x \cdot x \cdot y + 0 \cdot x \cdot y \cdot x + 0 \cdot x \cdot y \cdot y + 0 \cdot y \cdot x \cdot x + 0 \cdot y \cdot x \cdot y +$
$+ 0 \cdot y \cdot y \cdot x + 0 \cdot y \cdot y \cdot y = 0$

$f_{xxxx}(0;0) = 0 \quad f_{xxxx} = y^3 \cdot \mathrm{e}^{xy} \cdot y = y^4 \cdot \mathrm{e}^{xy}$
$f_{xxxy}(0;0) = 0 \quad f_{xxxy} = 3y^2 \cdot \mathrm{e}^{xy} + y^3 \cdot \mathrm{e}^{xy} \cdot x = 3y^2 \cdot \mathrm{e}^{xy} + xy^3 \cdot \mathrm{e}^{xy}$
$f_{xxyx}(0;0) = 0 \quad f_{xxyx} = 2y \cdot \mathrm{e}^{xy} \cdot y + y^2 \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy} \cdot y = 3y^2 \cdot \mathrm{e}^{xy} + xy^3 \cdot \mathrm{e}^{xy}$
$f_{xxyy}(0;0) = 2 \quad f_{xxyy} = 2 \cdot \mathrm{e}^{xy} + 2y \cdot \mathrm{e}^{xy} \cdot x + 2xy \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy} \cdot x =$
$= 2 \cdot \mathrm{e}^{xy} + 4xy \cdot \mathrm{e}^{xy} + x^2 y^2 \cdot \mathrm{e}^{xy}$
$f_{xyxx}(0;0) = 0 \quad f_{xyxx} = 2y \cdot \mathrm{e}^{xy} \cdot y + y^2 \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy} \cdot y = 3y^2 \cdot \mathrm{e}^{xy} + xy^3 \cdot \mathrm{e}^{xy}$
$f_{xyxy}(0;0) = 2 \quad f_{xyxy} = 2 \cdot \mathrm{e}^{xy} + 2y \cdot \mathrm{e}^{xy} \cdot x + 2xy \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy} \cdot x =$
$= 2 \cdot \mathrm{e}^{xy} + 4xy \cdot \mathrm{e}^{xy} + x^2 y^2 \cdot \mathrm{e}^{xy}$
$f_{xyyx}(0;0) = 2 \quad f_{xyyx} = 2 \cdot \mathrm{e}^{xy} + 2x \cdot \mathrm{e}^{xy} \cdot y + 2xy \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy} \cdot y =$
$= 2 \cdot \mathrm{e}^{xy} + 4xy \cdot \mathrm{e}^{xy} + x^2 y^2 \cdot \mathrm{e}^{xy}$
$f_{xyyy}(0;0) = 0 \quad f_{xyyy} = 2x \cdot \mathrm{e}^{xy} \cdot x + x^2 \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy} \cdot x = 3x^2 \cdot \mathrm{e}^{xy} + x^3 y \cdot \mathrm{e}^{xy}$

$f_{yxxx}(0;0)=0$ $\quad f_{yxxx}=2y\cdot e^{xy}\cdot y+y^2\cdot e^{xy}+xy^2\cdot e^{xy}\cdot y=3y^2\cdot e^{xy}+xy^3\cdot e^{xy}$

$f_{yxxy}(0;0)=2$ $\quad f_{yxxy}=2\cdot e^{xy}+2y\cdot e^{xy}\cdot x+2xy\cdot e^{xy}+xy^2\cdot e^{xy}\cdot x=$

$$=2\cdot e^{xy}+4xy\cdot e^{xy}+x^2y^2\cdot e^{xy}$$

$f_{yxyx}(0;0)=2$ $\quad f_{yxyx}=2\cdot e^{xy}+2x\cdot e^{xy}\cdot y+2xye^{xy}+x^2y\cdot e^{xy}\cdot y=$

$$=2\cdot e^{xy}+4xy\cdot e^{xy}+x^2y^2\cdot e^{xy}$$

$f_{yxyy}(0;0)=0$ $\quad f_{yxyy}=2x\cdot e^{xy}\cdot x+x^2\cdot e^{xy}+x^2y\cdot e^{xy}\cdot x=3x^2\cdot e^{xy}+x^3y\cdot e^{xy}$

$f_{yyxx}(0;0)=2$ $\quad f_{yyxx}=2\cdot e^{xy}+2x\cdot e^{xy}\cdot y+2xy\cdot e^{xy}+x^2y\cdot e^{xy}\cdot y=$

$$=2\cdot e^{xy}+4xy\cdot e^{xy}+x^2y^2\cdot e^{xy}$$

$f_{yyxy}(0;0)=0$ $\quad f_{yyxy}=2x\cdot e^{xy}\cdot x+x^2\cdot e^{xy}+x^2y\cdot e^{xy}\cdot x=3x^2\cdot e^{xy}+x^3y\cdot e^{xy}$

$f_{yyyx}(0;0)=0$ $\quad f_{yyyx}=3x^2\cdot e^{xy}+x^3\cdot e^{xy}\cdot y=3x^2\cdot e^{xy}+x^3y\cdot e^{xy}$

$f_{yyyy}(0;0)=0$ $\quad f_{yyyy}=x^3\cdot e^{xy}\cdot x=x^4\cdot e^{xy}$

$$\begin{aligned}d^4f(0;0)&=0\cdot x\cdot x\cdot x\cdot x+0\cdot x\cdot x\cdot x\cdot y+0\cdot x\cdot x\cdot y\cdot x+2\cdot x\cdot x\cdot y\cdot y+\\&+0\cdot x\cdot y\cdot x\cdot x+2\cdot x\cdot y\cdot x\cdot y+2\cdot x\cdot y\cdot y\cdot x+0\cdot x\cdot y\cdot y\cdot y+0\cdot y\cdot x\cdot x\cdot x+\\&+2\cdot y\cdot x\cdot x\cdot y+2\cdot y\cdot x\cdot y\cdot x+0\cdot y\cdot x\cdot y\cdot y+2\cdot y\cdot y\cdot x\cdot x+0\cdot y\cdot y\cdot x\cdot y+\\&+0\cdot y\cdot y\cdot y\cdot x+0\cdot y\cdot y\cdot y\cdot y=6\cdot(2x^2y^2)=\\&=12x^2y^2=\underbrace{4\cdot 3}_{12}\cdot\frac{2!}{2!}\cdot x^2y^2=\frac{4!}{2!}\cdot x^2y^2=4!\cdot\frac{x^2y^2}{2!}\end{aligned}$$

$f_{xxxxx}(0;0)=0$ $\quad f_{xxxxx}=y^4\cdot e^{xy}\cdot y=y^5\cdot e^{xy}$

$\vdots$

$$d^5f(0;0)=0\cdot x\cdot x\cdot x\cdot x\cdot x+\ldots=0$$

$$\underline{\underline{M=}}\quad f(0;0)+\frac{df(0;0)}{1!}+\frac{d^2f(0;0)}{2!}+\frac{d^3f(0;0)}{3!}+\cdots=1+0+\frac{2!\cdot\frac{xy}{1!}}{2!}+0+\frac{4!\cdot\frac{x^2y^2}{2!}}{4!}+\ldots=$$

$$\underline{\underline{=1+\frac{xy}{1!}+\frac{(xy)^2}{2!}+\frac{(xy)^3}{3!}+\frac{(xy)^4}{4!}+\frac{(xy)^5}{5!}+\frac{(xy)^6}{6!}+\cdots}}$$

8. [$x \in \mathbb{R}$; $y \in \mathbb{R}$] V okolí středu $C = [1; -1]$ rozviňte do **Taylorovy** řady funkci

$$f(x,y) = x^3 + xy^2 - 3x + 2xy + 1$$

$$f(1;-1) = 1^3 + 1 \cdot (-1)^2 - 3 \cdot 1 + 2 \cdot 1 \cdot (-1) + 1 = -2$$

$$\begin{array}{ll} f_x(1;-1) = -1 & f_x = 3x^2 + y^2 - 3 + 2y \\ f_y(1;-1) = 0 & f_y = 2xy + 2x \end{array}$$

$$\mathrm{d}f(1;-1) = -1 \cdot (x-1) + 0 \cdot (y+1) = -(x-1)$$

$$\begin{array}{ll} f_{xx}(1;-1) = 6 & f_{xx} = 6x \\ f_{xy}(1;-1) = 0 & f_{xy} = 2y + 2 \\ \hline f_{yx}(1;-1) = 0 & f_{yx} = 2y + 2 \\ f_{yy}(1;-1) = 2 & f_{yy} = 2x \end{array}$$

$$\mathrm{d}^2 f(1;-1) = 6 \cdot (x-1) \cdot (x-1) + 0 \cdot (x-1) \cdot (y+1) + 0 \cdot (y+1) \cdot (x-1) + \\ + 2 \cdot (y+1) \cdot (y+2) = 6(x-1)^2 + 2(y+1)^2$$

$$\begin{array}{ll} f_{xxx}(1;-1) = 6 & f_{xxx} = 6 \\ f_{xxy}(1;-1) = 0 & f_{xxy} = 0 \\ \hline f_{xyx}(1;-1) = 0 & f_{xyx} = 0 \\ f_{xyy}(1;-1) = 2 & f_{xyy} = 2 \\ \hline f_{yxx}(1;-1) = 0 & f_{yxx} = 0 \\ f_{yxy}(1;-1) = 2 & f_{yxy} = 2 \\ \hline f_{yyx}(1;-1) = 2 & f_{yyx} = 2 \\ f_{yyy}(1;-1) = 0 & f_{yyy} = 0 \end{array}$$

$$\mathrm{d}^3 f(1;-1) = 6 \cdot (x-1) \cdot (x-1) \cdot (x-1) + 0 + 0 + 2 \cdot (x-1) \cdot (y+1) \cdot (y+1) + 0 - \\ + 2 \cdot (y+1) \cdot (x-1) \cdot (y+1) + 2 \cdot (y+1) \cdot (y+1) \cdot (x-1) + 0 = \\ = 6(x-1)^3 + 6(x-1)(y+1)^2$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4 f(1;-1) = 0;\ \mathrm{d}^5 f(1;-1) = 0;\ \mathrm{d}^6 f(1;-1) = 0;\ \mathrm{d}^7 f(1;-1) = 0;\ \ldots \quad \text{proto}$$

$$\underline{\underline{T(C) =}} \quad f(1;-1) + \frac{\mathrm{d}f(1;-1)}{1!} + \frac{\mathrm{d}^2 f(1;-1)}{2!} + \frac{\mathrm{d}^3 f(1;-1)}{3!} =$$

$$\underline{\underline{-2 - (x-1) + 3(x-1)^2 + (y+1)^2 + (x-1)^3 + (x-1)(y+1)^2}}$$

9. $[x \in \mathbb{R};\ y \in \mathbb{R}]$ Rozviňte do **Maclaurinovy** řady funkci $f(x,y) = \cos(xy)$ (viz poznámka).

$f(0;0) = \cos(0 \cdot 0) = \cos 0 = 1$

$f_x(0;0) = 0 \quad f_x = -\sin(xy) \cdot y = -y \cdot \sin(xy)$
$f_y(0;0) = 0 \quad f_y = -\sin(xy) \cdot x = -x \cdot \sin(xy)$

$\mathrm{d}f(0;0) = 0 \cdot (x-0) + 0 \cdot (y-0) = 0$

$f_{xx}(0;0) = 0 \quad f_{xx} = -y \cdot \cos(xy) \cdot y = -y^2 \cdot \cos(xy)$
$f_{xy}(0;0) = 0 \quad f_{xy} = -1 \cdot \sin(xy) - y \cdot \cos(xy) \cdot x = -\sin(xy) - xy \cdot \cos(xy)$
$f_{yx}(0;0) = 0 \quad f_{yx} = -1 \cdot \sin(xy) - x \cdot \cos(xy) \cdot y = -\sin(xy) - xy \cdot \cos(xy)$
$f_{yy}(0;0) = 0 \quad f_{yy} = -x \cdot \cos(xy) \cdot x = -x^2 \cdot \cos(xy)$

$\mathrm{d}^2 f(0;0) = 0 \cdot x \cdot x + 0 \cdot x \cdot y + 0 \cdot y \cdot x + 0 \cdot y \cdot y = 0$

$f_{xxx}(0;0) = 0 \quad f_{xxx} = -y^2 \cdot [-\sin(xy) \cdot y] = y^3 \cdot \sin(xy)$
$f_{xxy}(0;0) = 0 \quad f_{xxy} = -2y \cdot \cos(xy) - y^2 \cdot [-\sin(xy) \cdot x] = xy^2 \cdot \sin(xy) - 2y \cdot \cos(xy)$
$f_{xyx}(0;0) = 0 \quad f_{xyx} = -\cos(xy) \cdot y - y \cdot \cos(xy) - xy \cdot [-\sin(xy) \cdot y] = xy^2 \cdot \sin(xy) - 2y \cdot \cos(xy)$
$f_{xyy}(0;0) = 0 \quad f_{xyy} = -\cos(xy) \cdot x - x \cdot \cos(xy) - xy \cdot [-\sin(xy) \cdot x] = x^2 y \cdot \sin(xy) - 2x \cdot \cos(xy)$
$f_{yxx}(0;0) = 0 \quad f_{yxx} = -\cos(xy) \cdot y - y \cdot \cos(xy) - xy \cdot [-\sin(xy) \cdot y] = xy^2 \cdot \sin(xy) - 2y \cdot \cos(xy)$
$f_{yxy}(0;0) = 0 \quad f_{yxy} = -\cos(xy) \cdot x - x \cdot \cos(xy) - xy \cdot [-\sin(xy) \cdot x] = x^2 y \cdot \sin(xy) - 2x \cdot \cos(xy)$
$f_{yyx}(0;0) = 0 \quad f_{yyx} = -2x \cdot \cos(xy) - x^2 \cdot [-\sin(xy) \cdot y] = x^2 y \cdot \sin(xy) - 2x \cdot \cos(xy)$
$f_{yyy}(0;0) = 0 \quad f_{yyy} = -x^2 \cdot [-\sin(xy) \cdot x] = x^3 \cdot \sin(xy)$

$\mathrm{d}^3 f(0;0) = 0 \cdot x \cdot x \cdot x + 0 \cdot x \cdot x \cdot y + 0 \cdot x \cdot y \cdot x + 0 \cdot x \cdot y \cdot y + 0 \cdot y \cdot x \cdot x + 0 \cdot y \cdot x \cdot y + 0 \cdot y \cdot y \cdot x + 0 \cdot y \cdot y \cdot y = 0$

$f_{xxxx}(0;0) = 0 \quad f_{xxxx} = y^3 \cdot \cos(xy) \cdot y = y^4 \cdot \cos(xy)$
$f_{xxxy}(0;0) = 0 \quad f_{xxxy} = 3y^2 \cdot \sin(xy) + y^3 \cdot \cos(xy) \cdot x = 3y^2 \cdot \sin(xy) + xy^3 \cdot \cos(xy)$
$f_{xxyx}(0;0) = 0 \quad f_{xxyx} = y^2 \cdot \sin(xy) + xy^2 \cdot \cos(xy) \cdot y - 2y \cdot [-\sin(xy) \cdot y] = 3y^2 \cdot \sin(xy) + xy^3 \cdot \cos(xy)$
$f_{xxyy}(0;0) = -2 \quad f_{xxyy} = 2xy \cdot \sin(xy) + xy^2 \cdot \cos(xy) \cdot x - 2 \cdot \cos(xy) - 2y \cdot [-\sin(xy) \cdot x] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$$f_{xyxx}(0;0)=0 \qquad f_{xyxx}=y^2\cdot\sin(xy)+xy^2\cdot\cos(xy)\cdot y-2y\cdot[-\sin(xy)\cdot y]=\\ =3y^2\cdot\sin(xy)+xy^3\cdot\cos(xy)$$

$$f_{xyxy}(0;0)=-2 \qquad f_{xyxy}=2xy\cdot\sin(xy)+xy^2\cos(xy)\cdot x-2\cos(xy)-\\ -2y\cdot[-\sin(xy)\cdot x]=4xy\cdot\sin(xy)+(x^2y^2-2)\cdot\cos(xy)$$

$$f_{xyyx}(0;0)=-2 \qquad f_{xyyx}=2xy\cdot\sin(xy)+x^2y\cdot\cos(xy)\cdot y-2\cdot\cos(xy)-\\ -2x\cdot[-\sin(xy)\cdot y]=4xy\cdot\sin(xy)+(x^2y^2-2)\cdot\cos(xy)$$

$$f_{xyyy}(0;0)=0 \qquad f_{xyyy}=x^2\cdot\sin(xy)+x^2y\cdot\cos(xy)\cdot x-2x\cdot[-\sin(xy)\cdot x]=\\ =3x^2\cdot\sin(xy)+x^3y\cdot\cos(xy)$$

$$f_{yxxx}(0;0)=0 \qquad f_{yxxx}=y^2\cdot\sin(xy)+xy^2\cdot\cos(xy)\cdot y-2y\cdot[-\sin(xy)\cdot y]=\\ =3y^2\cdot\sin(xy)+xy^3\cdot\cos(xy)$$

$$f_{yxxy}(0;0)=-2 \qquad f_{yxxy}=2xy\cdot\sin(xy)+xy^2\cdot\cos(xy)\cdot x-2\cdot\cos(xy)-\\ -2y\cdot[-\sin(xy)\cdot x]=4xy\cdot\sin(xy)+(x^2y^2-2)\cdot\cos(xy)$$

$$f_{yxyx}(0;0)=-2 \qquad f_{yxyx}=2xy\cdot\sin(xy)+x^2y\cdot\cos(xy)\cdot y-2\cdot\cos(xy)-\\ -2x\cdot[-\sin(xy)\cdot y]=4xy\cdot\sin(xy)+(x^2y^2-2)\cdot\cos(xy)$$

$$f_{yxyy}(0;0)=0 \qquad f_{yxyy}=x^2\cdot\sin(xy)+x^2y\cdot\cos(xy)\cdot x-2x\cdot[-\sin(xy)\cdot x]=\\ =3x^2\cdot\sin(xy)+x^3y\cdot\cos(xy)$$

$$f_{yyxx}(0;0)=-2 \qquad f_{yyxx}=2xy\cdot\sin(xy)+x^2y\cdot\cos(xy)\cdot y-2\cdot\cos(xy)-\\ -2x\cdot[-\sin(xy)\cdot y]=4xy\cdot\sin(xy)+(x^2y^2-2)\cdot\cos(xy)$$

$$f_{yyxy}(0;0)=0 \qquad f_{yyxy}=x^2\cdot\sin(xy)+x^2y\cdot\cos(xy)\cdot x-2x\cdot[-\sin(xy)\cdot x]=\\ =3x^2\cdot\sin(xy)+x^3y\cdot\cos(xy)$$

$$f_{yyyx}(0;0)=0 \qquad f_{yyyx}=3x^2\cdot\sin(xy)+x^3\cdot\cos(xy)\cdot y=\\ =3x^2\cdot\sin(xy)+x^3y\cdot\cos(xy)$$

$$f_{yyyy}(0;0)=0 \qquad f_{yyyy}=x^3\cdot\cos(xy)\cdot x=x^4\cdot\cos(xy)$$

$$\begin{aligned}\mathrm{d}^4f(0;0)&=0\cdot x\cdot x\cdot x\cdot x+0\cdot x\cdot x\cdot x\cdot y+0\cdot x\cdot x\cdot y\cdot x-2\cdot x\cdot x\cdot y\cdot y+\\ &+0\cdot x\cdot y\cdot x\cdot x-2\cdot x\cdot y\cdot x\cdot y-2\cdot x\cdot y\cdot y\cdot x+0\cdot x\cdot y\cdot y\cdot y+0\cdot y\cdot x\cdot x\cdot x+\\ &-2\cdot y\cdot x\cdot x\cdot y-2\cdot y\cdot x\cdot y\cdot x+0\cdot y\cdot x\cdot y\cdot y-2\cdot y\cdot y\cdot x\cdot x+0\cdot y\cdot y\cdot x\cdot y+\\ &+0\cdot y\cdot y\cdot y\cdot x+0\cdot y\cdot y\cdot y\cdot y=6\cdot(-2x^2y^2)=\\ &=-12x^2y^2=-\underbrace{4\cdot 3}_{12}\cdot\frac{2!}{2!}\cdot x^2y^2=-\frac{4!}{2!}\cdot x^2y^2=-4!\cdot\frac{x^2y^2}{2!}\end{aligned}$$

$$f_{xxxxx}(0;0)=0 \qquad f_{xxxxx}=y^4\cdot\cos(xy)\cdot y=y^5\cdot\cos(xy)$$

$$\vdots$$

$$\mathrm{d}^5f(0;0)=0\cdot x\cdot x\cdot x\cdot x\cdot x+\ldots=0$$

$$\underline{\underline{M=}}\quad f(0;0)+\frac{\mathrm{d}f(0;0)}{1!}+\frac{\mathrm{d}^2f(0;0)}{2!}+\frac{\mathrm{d}^3f(0;0)}{3!}+\cdots=1+0+0+0+\frac{-4!\cdot\frac{x^2y^2}{2!}}{4!}+0+\ldots=$$

$$\underline{\underline{=1-\frac{(xy)^2}{2!}+\frac{(xy)^4}{4!}-\frac{(xy)^6}{6!}+\cdots}}$$

Funkce cos je SUDÁ.

10. $[x \in \mathbb{R};\ y \in \mathbb{R}]$ V okolí středu $D = [-1; 1]$ rozviňte do **Taylorovy** řady funkci

$$f(x,y) = y^3 + x^2y + 2xy - 3y + 1$$

$$f(-1;1) = 1^3 + (-1)^2 \cdot 1 + 2 \cdot (-1) \cdot 1 - 3 \cdot 1 + 1 = -2$$

$$\begin{array}{ll} f_x(-1;1) = 0 & f_x = 2xy + 2y \\ f_y(-1;1) = -1 & f_y = 3y^2 + x^2 + 2x - 3 \end{array}$$

$$\mathrm{d}f(-1;1) = 0 \cdot (x+1) - 1 \cdot (y-1) = -(y-1)$$

$$\begin{array}{ll} f_{xx}(-1;1) = 2 & f_{xx} = 2y \\ f_{xy}(-1;1) = 0 & f_{xy} = 2x + 2 \\ \hline f_{yx}(-1;1) = 0 & f_{yx} = 2x + 2 \\ f_{yy}(-1;1) = 6 & f_{yy} = 6y \end{array}$$

$$\mathrm{d}^2f(-1;1) = 2 \cdot (x+1) \cdot (x+1) + 0 \cdot (x+1) \cdot (y-1) + 0 \cdot (y-1) \cdot (x+1) + \\ + 6 \cdot (y-1) \cdot (y-1) = 2(x+1)^2 + 6(y-1)^2$$

$$\begin{array}{ll} f_{xxx}(-1;1) = 0 & f_{xxx} = 0 \\ f_{xxy}(-1;1) = 2 & f_{xxy} = 2 \\ \hline f_{xyx}(-1;1) = 2 & f_{xyx} = 2 \\ f_{xyy}(-1;1) = 0 & f_{xyy} = 0 \\ \hline f_{yxx}(-1;1) = 2 & f_{yxx} = 2 \\ f_{yxy}(-1;1) = 0 & f_{yxy} = 0 \\ \hline f_{yyx}(-1;1) = 0 & f_{yyx} = 0 \\ f_{yyy}(-1;1) = 6 & f_{yyy} = 6 \end{array}$$

$$\mathrm{d}^3f(-1;1) = 0 \cdot (x+1) \cdot (x+1) \cdot (x+1) + 2 \cdot (x+1) \cdot (x+1) \cdot (y-1) \\ + 2 \cdot (x+1) \cdot (y-1) \cdot (x+1) + 0 + 2 \cdot (y-1) \cdot (x+1) \cdot (x+1) + 0 + 0 + \\ + 6 \cdot (y-1) \cdot (y-1) \cdot (y-1) = 6(x+1)^2(y-1) + 6(y-1)^3$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4f(-1;1) = 0;\ \mathrm{d}^5f(-1;1) = 0;\ \mathrm{d}^6f(-1;1) = 0;\ \mathrm{d}^7f(-1;1) = 0;\ \dots \quad \text{proto}$$

$$\underline{\underline{T(D) =}} \quad f(-1;1) + \frac{\mathrm{d}f(-1;1)}{1!} + \frac{\mathrm{d}^2f(-1;1)}{2!} + \frac{\mathrm{d}^3f(-1;1)}{3!} = \\ \underline{\underline{-2 - (y-1) + (x+1)^2 + 3(y-1)^2 + (x+1)^2(y-1) + (y-1)^3}}$$

11. [$x \in \mathbb{R}\,;\ y \in \mathbb{R}$] Rozviňte do **Maclaurinovy** řady funkci $f(x,y) = \mathrm{e}^{x^2+y^2}$ (viz poznámka).

$$f(0;0) = \mathrm{e}^{0^2+0^2} = \mathrm{e}^0 = 1$$

$$f_x(0;0) = 0 \quad f_x = \mathrm{e}^{x^2+y^2} \cdot 2x = 2x \cdot \mathrm{e}^{x^2+y^2}$$
$$f_y(0;0) = 0 \quad f_y = \mathrm{e}^{x^2+y^2} \cdot 2y = 2y \cdot \mathrm{e}^{x^2+y^2}$$

$$\mathrm{d}f(0;0) = 0 \cdot (x-0) + 0 \cdot (y-0) = 0$$

$$f_{xx}(0;0) = 2 \quad f_{xx} = 2\mathrm{e}^{x^2+y^2} + 2x\mathrm{e}^{x^2+y^2} \cdot 2x = (2+4x^2) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xy}(0;0) = 0 \quad f_{xy} = 2x\mathrm{e}^{x^2+y^2} \cdot 2y = 4xy \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{yx}(0;0) = 0 \quad f_{yx} = 2y\mathrm{e}^{x^2+y^2} \cdot 2x = 4xy \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{yy}(0;0) = 2 \quad f_{yy} = 2\mathrm{e}^{x^2+y^2} + 2y\mathrm{e}^{x^2+y^2} \cdot 2y = (2+4y^2) \cdot \mathrm{e}^{x^2+y^2}$$

$$\mathrm{d}^2 f(0;0) = 2x^2 + 2y^2 = 2 \cdot (x^2+y^2) = 2 \cdot \frac{1!}{1!} \cdot (x^2+y^2) = \frac{2!}{1!} \cdot (x^2+y^2) = 2! \cdot \frac{x^2+y^2}{1!}$$

$$f_{xxx}(0;0) = 0 \quad f_{xxx} = 8x\mathrm{e}^{x^2+y^2} + (2+4x^2)\mathrm{e}^{x^2+y^2} \cdot 2x = (12x+8x^3) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xxy}(0;0) = 0 \quad f_{xxy} = (2+4x^2)\mathrm{e}^{x^2+y^2} \cdot 2y = (4y+8x^2y) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xyx}(0;0) = 0 \quad f_{xyx} = 4y\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2x = (4y+8x^2y) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xyy}(0;0) = 0 \quad f_{xyy} = 4x\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2y = (4x+8xy^2) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{yxx}(0;0) = 0 \quad f_{yxx} = 4y\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2x = (4y+8x^2y) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{yxy}(0;0) = 0 \quad f_{yxy} = 4x\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2y = (4x+8xy^2) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{yyx}(0;0) = 0 \quad f_{yyx} = (2+4y^2)\mathrm{e}^{x^2+y^2} \cdot 2x = (4x+8xy^2) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{yyy}(0;0) = 0 \quad f_{yyy} = 8y\mathrm{e}^{x^2+y^2} + (2+4y^2)\mathrm{e}^{x^2+y^2} \cdot 2y = (12y+8y^3) \cdot \mathrm{e}^{x^2+y^2}$$

$$\mathrm{d}^3 f(0;0) = 0$$

$$f_{xxxx}(0;0) = 12 \quad f_{xxxx} = (12+24x^2)\mathrm{e}^{x^2+y^2} + (12x+8x^3)\mathrm{e}^{x^2+y^2} \cdot 2x = (12+48x^2+16x^4) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xxxy}(0;0) = 0 \quad f_{xxxy} = (12x+8x^3)\mathrm{e}^{x^2+y^2} \cdot 2y = (24xy+16x^3y) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xxyx}(0;0) = 0 \quad f_{xxyx} = 16xy\mathrm{e}^{x^2+y^2} + (4y+8x^2y)\mathrm{e}^{x^2+y^2} \cdot 2x = (24xy+16x^3y) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xxyy}(0;0) = 4 \quad f_{xxyy} = (4+8x^2)\mathrm{e}^{x^2+y^2} + (4y+8x^2y)\mathrm{e}^{x^2+y^2} \cdot 2y = (4+8x^2+8y^2+16x^2y^2) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xyxx}(0;0) = 0 \quad f_{xyxx} = 16xy\mathrm{e}^{x^2+y^2} + (4y+8x^2y)\mathrm{e}^{x^2+y^2} \cdot 2x = (24xy+16x^3y) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xyxy}(0;0) = 4 \quad f_{xyxy} = (4+8x^2)\mathrm{e}^{x^2+y^2} + (4y+8x^2y)\mathrm{e}^{x^2+y^2} \cdot 2y = (4+8x^2+8y^2+16x^2y^2) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xyyx}(0;0) = 4 \quad f_{xyyx} = (4+8y^2)\mathrm{e}^{x^2+y^2} + (4x+8xy^2)\mathrm{e}^{x^2+y^2} \cdot 2x = (4+8x^2+8y^2+16x^2y^2) \cdot \mathrm{e}^{x^2+y^2}$$
$$f_{xyyy}(0;0) = 0 \quad f_{xyyy} = 16xy\mathrm{e}^{x^2+y^2} + (4x+8xy^2)\mathrm{e}^{x^2+y^2} \cdot 2y = (24xy+16xy^3) \cdot \mathrm{e}^{x^2+y^2}$$

$f_{yxxx}(0;0)=0$ $\quad f_{yxxx}=16xye^{x^2+y^2}+(4y+8x^2y)e^{x^2+y^2}\cdot 2x=$

$$=(24xy+16x^3y)\cdot e^{x^2+y^2}$$

$f_{yxxy}(0;0)=4$ $\quad f_{yxxy}=(4+8x^2)e^{x^2+y^2}+(4y+8x^2y)e^{x^2+y^2}\cdot 2y=$

$$=(4+8x^2+8y^2+16x^2y^2)\cdot e^{x^2+y^2}$$

$f_{yxyx}(0;0)=4$ $\quad f_{yxyx}=(4+8y^2)e^{x^2+y^2}+(4x+8xy^2)e^{x^2+y^2}\cdot 2x=$

$$=(4+8x^2+8y^2+16x^2y^2)\cdot e^{x^2+y^2}$$

$f_{yxyy}(0;0)=0$ $\quad f_{yxyy}=16xye^{x^2+y^2}+(4x+8xy^2)e^{x^2+y^2}\cdot 2y=$

$$=(24xy+16xy^3)\cdot e^{x^2+y^2}$$

$f_{yyxx}(0;0)=4$ $\quad f_{yyxx}=(4+8y^2)e^{x^2+y^2}+(4x+8xy^2)e^{x^2+y^2}\cdot 2x=$

$$=(4+8x^2+8y^2+16x^2y^2)\cdot e^{x^2+y^2}$$

$f_{yyxy}(0;0)=0$ $\quad f_{yyxy}=16xye^{x^2+y^2}+(4x+8xy^2)e^{x^2+y^2}\cdot 2y=$

$$=(24xy+16xy^3)\cdot e^{x^2+y^2}$$

$f_{yyyx}(0;0)=0$ $\quad f_{yyyx}=(12y+8y^3)e^{x^2+y^2}\cdot 2x=(24xy+16xy^3)\cdot e^{x^2+y^2}$

$f_{yyyy}(0;0)=12$ $\quad f_{yyyy}=(12+24y^2)e^{x^2+y^2}+(12y+8y^3)e^{x^2+y^2}\cdot 2y=$

$$=(12+48y^2+16y^4)\cdot e^{x^2+y^2}$$

$$\begin{aligned}\mathrm{d}^4f(0;0)&=12xxxx+0+0+4xxyy+0+4xyxy+4xyyx+0+0+4yxxy+4yxyx+0+\\&+4yyxx+0+0+12yyyy=\\&=12(x^4+2x^2y^2+y^4)=\underbrace{4\cdot 3}_{12}\cdot\frac{2!}{2!}\cdot(x^2+y^2)^2=\frac{4!}{2!}\cdot(x^2+y^2)^2\end{aligned}$$

$$\underline{\underline{M=}}\quad f(0;0)+\frac{\mathrm{d}f(0;0)}{1!}+\frac{\mathrm{d}^2f(0;0)}{2!}+\frac{\mathrm{d}^3f(0;0)}{3!}+\frac{\mathrm{d}^4f(0;0)}{4!}+\cdots=$$

$$=1+0+\frac{2!\cdot\frac{(x^2+y^2)}{1!}}{2!}+0+\frac{4!\cdot\frac{(x^2+y^2)^2}{2!}}{4!}+0+\frac{6!\cdot\frac{(x^2+y^2)^3}{3!}}{6!}+\ldots=$$

$$\underline{\underline{=1+\frac{(x^2+y^2)}{1!}+\frac{(x^2+y^2)^2}{2!}+\frac{(x^2+y^2)^3}{3!}+\frac{(x^2+y^2)^4}{4!}+\frac{(x^2+y^2)^5}{5!}+\cdots}}$$

12. $[x \in \mathbb{R}\,;\ y \in \mathbb{R}]$ V okolí středu $E = [1; -3]$ rozviňte do **Taylorovy** řady funkci

$$f(x, y) = x^3 + xy - y - 1$$

$f(1; -3) = 1^3 + 1 \cdot (-3) - (-3) - 1 = 0$

$f_x(1; -3) = 0 \quad f_x = 3x^2 + y$
$f_y(1; -3) = 0 \quad f_y = x - 1$

$\mathrm{d}f(1; -3) = 0 \cdot (x - 1) + 0 \cdot (y + 3) = 0$

$f_{xx}(1; -3) = 6 \quad f_{xx} = 6x$
$f_{xy}(1; -3) = 1 \quad f_{xy} = 1$
$f_{yx}(1; -3) = 1 \quad f_{yx} = 1$
$f_{yy}(1; -3) = 0 \quad f_{yy} = 0$

$$\mathrm{d}^2 f(1; -3) = 6 \cdot (x - 1) \cdot (x - 1) + 1 \cdot (x - 1) \cdot (y + 3) + 1 \cdot (y + 3) \cdot (x - 1) + \\ + 0 \cdot (y + 3) \cdot (y + 3) = 6(x - 1)^2 + 2(y + 3)$$

$f_{xxx}(1; -3) = 6 \quad f_{xxx} = 6$
$f_{xxy}(1; -3) = 0 \quad f_{xxy} = 0$
$f_{xyx}(1; -3) = 0 \quad f_{xyx} = 0$
$f_{xyy}(1; -3) = 0 \quad f_{xyy} = 0$
$f_{yxx}(1; -3) = 0 \quad f_{yxx} = 0$
$f_{yxy}(1; -3) = 0 \quad f_{yxy} = 0$
$f_{yyx}(1; -3) = 0 \quad f_{yyx} = 0$
$f_{yyy}(1; -3) = 0 \quad f_{yyy} = 0$

$\mathrm{d}^3 f(1; -3) = 6 \cdot (x - 1) \cdot (x - 1) \cdot (x - 1) + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 6(x - 1)^3$

Všechny další parciální derivace jsou rovny nule,

$\mathrm{d}^4 f(1; -3) = 0;\ \mathrm{d}^5 f(1; -3) = 0;\ \mathrm{d}^6 f(1; -3) = 0;\ \mathrm{d}^7 f(1; -3) = 0;\ \ldots$ proto

$$\underline{\underline{T(E) =}} \quad f(1; -3) + \frac{\mathrm{d}f(1; -3)}{1!} + \frac{\mathrm{d}^2 f(1; -3)}{2!} + \frac{\mathrm{d}^3 f(1; -3)}{3!} =$$

$$\underline{\underline{3(x - 1)^2 + (x - 1)(y + 3) + (x - 1)^3}}$$

13. $[x \in \mathbb{R}\,;\ y \in \mathbb{R}]$ Rozviňte do **Maclaurinovy** řady funkci

$$f(x,y) = \sin(x^2 + y^2)$$

$f(0;0) = \sin(0^2 + 0^2) = \sin 0 = 0$

$f_x(0;0) = 0 \quad f_x = \cos(x^2 + y^2) \cdot 2x = 2x \cdot \cos(x^2 + y^2)$
$f_y(0;0) = 0 \quad f_y = \cos(x^2 + y^2) \cdot 2y = 2y \cdot \cos(x^2 + y^2)$

$\mathrm{d}f(0;0) = 0 \cdot (x - 0) + 0 \cdot (y - 0) = 0 \cdot x + 0 \cdot y = 0$

$f_{xx}(0;0) = 2 \quad f_{xx} = 2 \cdot \cos(x^2 + y^2) - 2x \cdot \sin(x^2 + y^2) \cdot 2x =$
$= 2 \cdot \cos(x^2 + y^2) - 4x^2 \cdot \sin(x^2 + y^2)$
$f_{xy}(0;0) = 0 \quad f_{xy} = -2x \cdot \sin(x^2 + y^2) \cdot 2y = -4xy \cdot \sin(x^2 + y^2)$
$f_{yx}(0;0) = 0 \quad f_{yx} = -2y \cdot \sin(x^2 + y^2) \cdot 2x = -4xy \cdot \sin(x^2 + y^2)$
$f_{yy}(0;0) = 2 \quad f_{yy} = 2 \cdot \cos(x^2 + y^2) - 2y \cdot \sin(x^2 + y^2) \cdot 2y =$
$= 2 \cdot \cos(x^2 + y^2) - 4y^2 \cdot \sin(x^2 + y^2)$

$\mathrm{d}^2 f(0;0) = 2 \cdot x^2 + 0 \cdot xy + 0 \cdot yx + 2 \cdot y^2 = 2 \cdot (x^2 + y^2) = \dfrac{2}{1!} \cdot (x^2 + y^2)$

$f_{xxx}(0;0) = 0 \quad f_{xxx} = -2 \cdot \sin(x^2 + y^2) \cdot 2x - 8x \cdot \sin(x^2 + y^2) -$
$-4x^2 \cdot \cos(x^2 + y^2) \cdot 2x = -12x \cdot \sin(x^2 + y^2) - 8x^3 \cdot \cos(x^2 + y^2)$
$f_{xxy}(0;0) = 0 \quad f_{xxy} = -2 \cdot \sin(x^2 + y^2) \cdot 2y - 4x^2 \cdot \cos(x^2 + y^2) \cdot 2y =$
$= -4y \cdot \sin(x^2 + y^2) - 8x^2 y \cdot \cos(x^2 + y^2)$
$f_{xyx}(0;0) = 0 \quad f_{xyx} = -4y \cdot \sin(x^2 + y^2) - 4xy \cdot \cos(x^2 + y^2) \cdot 2x =$
$= -4y \cdot \sin(x^2 + y^2) - 8x^2 y \cdot \cos(x^2 + y^2)$
$f_{xyy}(0;0) = 0 \quad f_{xyy} = -4x \cdot \sin(x^2 + y^2) - 4xy \cdot \cos(x^2 + y^2) \cdot 2y =$
$= -4x \cdot \sin(x^2 + y^2) - 8xy^2 \cdot \cos(x^2 + y^2)$
$f_{yxx}(0;0) = 0 \quad f_{yxx} = -4y \cdot \sin(x^2 + y^2) - 4xy \cdot \cos(x^2 + y^2) \cdot 2x =$
$= -4y \cdot \sin(x^2 + y^2) - 8x^2 y \cdot \cos(x^2 + y^2)$
$f_{yxy}(0;0) = 0 \quad f_{yxy} = -4x \cdot \sin(x^2 + y^2) - 4xy \cdot \cos(x^2 + y^2) \cdot 2y =$
$= -4x \cdot \sin(x^2 + y^2) - 8xy^2 \cdot \cos(x^2 + y^2)$
$f_{yyx}(0;0) = 0 \quad f_{yyx} = -2 \cdot \sin(x^2 + y^2) \cdot 2x - 4y^2 \cdot \cos(x^2 + y^2) \cdot 2x =$
$= -4x \cdot \sin(x^2 + y^2) - 8xy^2 \cdot \cos(x^2 + y^2)$
$f_{yyy}(0;0) = 0 \quad f_{yyy} = -2 \cdot \sin(x^2 + y^2) \cdot 2y - 8y \cdot \sin(x^2 + y^2) -$
$-4y^2 \cdot \cos(x^2 + y^2) \cdot 2y = -12y \cdot \sin(x^2 + y^2) - 8y^3 \cdot \cos(x^2 + y^2)$

$\mathrm{d}^3 f(0;0) = 0 \cdot x^3 + 0 \cdot x^2 y + 0 \cdot xyx + 0 \cdot xy^2 + 0 \cdot yx^2 + 0 \cdot yxy + 0 \cdot y^2 x + 0 \cdot y^3 = 0$

$f_{xxxx}(0;0)=0$ $\quad f_{xxxx}=-12\cdot\sin(x^2+y^2)-12x\cdot\cos(x^2+y^2)\cdot 2x-$
$-24x^2\cdot\cos(x^2+y^2)+\;8x^3\cdot\sin(x^2+y^2)\cdot 2x=$
$=(16x^4-12)\cdot\sin(x^2+y^2)-48x^2\cdot\cos(x^2+y^2)$

$f_{xxxy}(0;0)=0$ $\quad f_{xxxy}=-12x\cdot\cos(x^2+y^2)\cdot 2y+8x^3\cdot\sin(x^2+y^2)\cdot 2y=$
$=16x^3y\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{xxyx}(0;0)=0$ $\quad f_{xxyx}=-4y\cdot\cos(x^2+y^2)\cdot 2x-16xy\cdot\cos(x^2+y^2)+$
$+8x^2y\cdot\sin(x^2+y^2)\cdot 2x=16x^3y\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{xxyy}(0;0)=0$ $\quad f_{xxyy}=-4\cdot\sin(x^2+y^2)-4y\cdot\cos(x^2+y^2)\cdot 2y-$
$-8x^2\cdot\cos(x^2+y^2)+8x^2y\sin(x^2+y^2)\cdot 2y=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{xyxx}(0;0)=0$ $\quad f_{xyxx}=-4y\cdot\cos(x^2+y^2)\cdot 2y-16xy\cdot\cos(x^2+y^2)+$
$+8x^2y\cdot\sin(x^2+y^2)\cdot 2x=16x^3y\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{xyxy}(0;0)=0$ $\quad f_{xyxy}=-4\cdot\sin(x^2+y^2)-4y\cdot\cos(x^2+y^2)\cdot 2y-$
$-8x^2\cdot\cos(x^2+y^2)+8x^2y\cdot\sin(x^2+y^2)\cdot 2y=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{xyyx}(0;0)=0$ $\quad f_{xyyx}=-4\cdot\sin(x^2+y^2)-4x\cdot\cos(x^2+y^2)\cdot 2x-$
$-8y^2\cdot\cos(x^2+y^2)+8xy^2\cdot\sin(x^2+y^2)\cdot 2x=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{xyyy}(0;0)=0$ $\quad f_{xyyy}=-4x\cdot\cos(x^2+y^2)\cdot 2y-16xy\cdot\cos(x^2+y^2)+$
$+8xy^2\cdot\sin(x^2+y^2)\cdot 2y=16xy^3\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yxxx}(0;0)=0$ $\quad f_{yxxx}=-4y\cdot\cos(x^2+y^2)\cdot 2x-16xy\cdot\cos(x^2+y^2)+$
$+8x^2y\cdot\sin(x^2+y^2)\cdot 2x=16x^3y\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yxxy}(0;0)=0$ $\quad f_{yxxy}=-4\cdot\sin(x^2+y^2)-4y\cdot\cos(x^2+y^2)\cdot 2y-$
$-8x^2\cdot\cos(x^2+y^2)+8x^2y\cdot\sin(x^2+y^2)\cdot 2y=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{yxyx}(0;0)=0$ $\quad f_{yxyx}=-4\cdot\sin(x^2+y^2)-4x\cdot\cos(x^2+y^2)\cdot 2x-$
$-8y^2\cdot\cos(x^2+y^2)+8xy^2\cdot\sin(x^2+y^2)\cdot 2x=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{yxyy}(0;0)=0$ $\quad f_{yxyy}=-4x\cdot\cos(x^2+y^2)\cdot 2y-16xy\cos(x^2+y^2)+$
$+8xy^2\cdot\sin(x^2+y^2)\cdot 2y=16xy^3\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yyxx}(0;0)=0$ $\quad f_{yyxx}=-4\cdot\sin(x^2+y^2)-4x\cdot\cos(x^2+y^2)\cdot 2x-$
$-8y^2\cdot\cos(x^2+y^2)+8xy^2\cdot\sin(x^2+y^2)\cdot 2x=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{yyxy}(0;0)=0$ $\quad f_{yyxy}=-4x\cdot\cos(x^2+y^2)\cdot 2y-16xy\cdot\cos(x^2+y^2)+$
$+8xy^2\cdot\sin(x^2+y^2)\cdot 2y=16xy^3\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yyyx}(0;0)=0$ $\quad f_{yyyx}=-12y\cdot\cos(x^2+y^2)\cdot 2x+8y^3\cdot\sin(x^2+y^2)\cdot 2x=$
$=16xy^3\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yyyy}(0;0)=0$ $\quad f_{yyyy}=-12\cdot\sin(x^2+y^2)-12y\cdot\cos(x^2+y^2)\cdot 2y-$
$-24y^2\cdot\cos(x^2+y^2)+8y^3\cdot\sin(x^2+y^2)\cdot 2y=$
$=(16y^4-12)\cdot\sin(x^2+y^2)-48y^2\cos(x^2+y^2)$

$$\mathrm{d}^4 f(0;0)=0\cdot x^4+4\cdot 0\cdot x^3y+6\cdot 0\cdot x^2y^2+4\cdot 0\cdot xy^3+0\cdot y^4=0$$

$f_{xxxxx}(0;0) = 0 \quad f_{xxxxx} = 64x^3 \cdot \sin(x^2+y^2) + (16x^4 - 12) \cdot \cos(x^2+y^2) \cdot 2x -$
$-96x \cdot \cos(x^2+y^2) + 48x^2 \cdot \sin(x^2+y^2) \cdot 2x =$
$= 160x^3 \cdot \sin(x^2+y^2) + (32x^5 - 120x) \cdot \cos(x^2+y^2)$

$f_{xxxxy}(0;0) = 0 \quad \ldots$

$\vdots$

$$\mathrm{d}^5 f(0;0) = 0 \cdot x^5 + 0 \cdot x^4 y + \ldots = 0$$

$f_{xxxxxx}(0;0) = -120 \quad f_{xxxxxx} = 480x^2 \cdot \sin(x^2+y^2) + 160x^3 \cdot \cos(x^2+y^2) \cdot 2x +$
$+(160x^4 - 120) \cdot \cos(x^2+y^2) -$
$-(32x^5 - 120x) \sin(x^2+y^2) \cdot 2x$

$f_{xxxxxy}(0;0) = 0 \quad \ldots$

$\vdots$

$$\mathrm{d}^6 f(0;0) = -120 \cdot x^6 + 0 \cdot x^5 \cdot y + \ldots - 40 \cdot x^4 \cdot y^2 + \ldots - 40 \cdot x^3 \cdot y \cdot x \cdot y + \ldots - 120 \cdot y^6 =$$
$$= -120x^6 - 360x^4y^2 - 360x^2y^4 - 120y^6 = -120 \cdot (x^6 + 3x^4y^2 + 3x^2y^4 + y^6) =$$
$$= -\underbrace{6 \cdot 5 \cdot 4}_{120} \cdot \frac{3!}{3!} \cdot (x^2+y^2)^3$$

$\vdots$

$$\underline{\underline{M =}} \quad f(0;0) + \frac{\mathrm{d}f(0;0)}{1!} + \frac{\mathrm{d}^2 f(0;0)}{2!} + \frac{\mathrm{d}^3 f(0;0)}{3!} + \cdots =$$
$$= 0 + 0 + \frac{2! \cdot \frac{(x^2+y^2)}{1!}}{2!} + 0 + 0 + 0 + \frac{-6! \cdot \frac{(x^2+y^2)^3}{3!}}{6!} + 0 + \ldots =$$
$$\underline{\underline{= \frac{(x^2+y^2)}{1!} - \frac{(x^2+y^2)^3}{3!} + \frac{(x^2+y^2)^5}{5!} - \frac{(x^2+y^2)^7}{7!} + \frac{(x^2+y^2)^9}{9!} - \cdots}}$$

Poznámka

Rozvineme pro $t = 0$ funkci (jedné proměnné) $\sin t$ a do výsledku dosadíme za

$$t = x^2 + y^2 .$$

f	$= \sin t$	$f(0)$	$=$	0
f'	$= \cos t$	$f'(0)$	$=$	1
f''	$= -\sin t$	$f''(0)$	$=$	0
$f^{(3)}$	$= -\cos t$	$f^{(3)}(0)$	$=$	-1
$f^{(4)}$	$= \sin t$	$f^{(4)}(0)$	$=$	0
$f^{(5)}$	$= \cos t$	$f^{(5)}(0)$	$=$	1
$f^{(6)}$	$= -\sin t$	$f^{(6)}(0)$	$=$	0
$f^{(7)}$	$= -\cos t$	$t^{(7)}(0)$	$=$	-1
		$\vdots$		

$$\sin t = \frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots$$