

# 1. Najděte neurčitý integrál $\int \sin x \cos x \, dx$ $D(f) = \mathbb{R}$

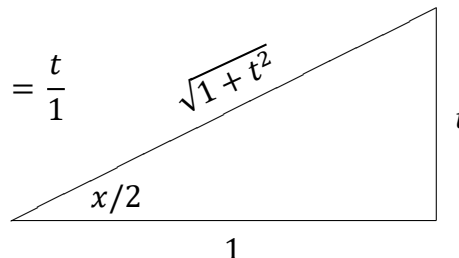
Zde je několik způsobů řešení:

## 1.1. Substitute $\operatorname{tg} \frac{x}{2} = t$

V pravoúhlém trojúhelníku pro úhel  $\frac{x}{2}$  je:  $\operatorname{tg} \frac{x}{2} = \frac{t}{1}$

Tím máme stanoveny velikosti obou odvěsen.

Velikost přepony určíme podle Pythagorovy věty.



$$\int \sin x \cos x \, dx = \left[ \begin{array}{ll} \operatorname{tg} \frac{x}{2} = t & x = 2 \operatorname{arctg} t^1 \\ \alpha = \frac{x}{2} & \frac{dx}{dt} = \frac{2}{1+t^2} \quad \left[ \frac{1}{\cos^2 \frac{x}{2}} \cdot \frac{1}{2} = t' = \frac{dt}{dx} \right] \\ \sin 2\alpha = 2 \sin \alpha \cos \alpha & \underline{\sin x} = 2 \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2} \\ \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha & \underline{\cos x} = \left( \frac{1}{\sqrt{1+t^2}} \right)^2 - \left( \frac{t}{\sqrt{1+t^2}} \right)^2 = \frac{1-t^2}{1+t^2} \end{array} \right]$$

$$= \int \frac{2t}{1+t^2} \cdot \frac{1-t^2}{1+t^2} \cdot \frac{2}{1+t^2} \, dt = \int \left[ \frac{At+B}{1+t^2} + \frac{Ct+D}{(1+t^2)^2} + \frac{Et+F}{(1+t^2)^3} \right] \, dt =$$

$$2t \cdot (1-t^2) \cdot 2 = (At+B) \cdot (1+t^2)^2 + (Ct+D) \cdot (1+t^2) + Et+F$$

$$4t - 4t^3 = (At+B) \cdot (1+2t^2+t^4) + (Ct+D) \cdot (1+t^2) + Et+F$$

$$t^5 : \quad 0 = A$$

$$t^4 : \quad 0 = B$$

$$t^3 : \quad -4 = 2A + C \quad \begin{array}{l} A=0 \\ \Rightarrow \end{array} \quad C = -4$$

$$t^2 : \quad 0 = 2B + D \quad \begin{array}{l} B=0 \\ \Rightarrow \end{array} \quad D = 0$$

$$t : \quad 4 = A + C + E \quad \begin{array}{l} A=0; C=-4 \\ \Rightarrow \end{array} \quad E = 8$$

$$t = 0 : \quad 0 = B + D + F \quad \begin{array}{l} B=0; D=0 \\ \Rightarrow \end{array} \quad F = 0$$

$$= \int \frac{-2}{(1+t^2)^2} 2t \, dt + \int \frac{4}{(1+t^2)^3} 2t \, dt = \left| \begin{array}{l} y = 1+t^2 \\ dy = 2t \, dt \end{array} \right| =$$

<sup>1</sup> Pokud  $\operatorname{tg} \frac{x}{2} = t$  a požadujeme vyjádřit  $x$  (v závislosti na  $t$ ), tak funkce  $\operatorname{tg} \frac{x}{2}$  není prostá na celém svém definičním oboru.

Proto k ní neexistuje inverzní funkce. Pro každý „kousek“ je inverzní funkce jiná (viz pravý sloupec), ale derivace

$$\frac{dx}{dt} = \frac{2}{1+t^2} \quad \text{je stejná pro libovolné } x \quad (\forall x).$$

$$\frac{x}{2} \in \left( \frac{-3\pi}{2}; \frac{-\pi}{2} \right) : \quad \frac{x}{2} = \operatorname{arctg} t - \pi$$

$$\frac{x}{2} \in \left( \frac{-\pi}{2}; \frac{\pi}{2} \right) : \quad \frac{x}{2} = \operatorname{arctg} t$$

$$\frac{x}{2} \in \left( \frac{\pi}{2}; \frac{3\pi}{2} \right) : \quad \frac{x}{2} = \operatorname{arctg} t + \pi$$

atd.

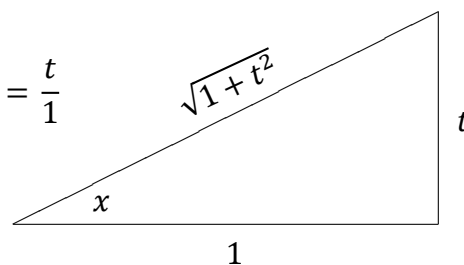
$$\begin{aligned}
&= \int \frac{-2}{y^2} dy + \int \frac{4}{y^3} dy = \int -2y^{-2} dy + \int 4y^{-3} dy = -2 \frac{y^{-1}}{-1} + 4 \frac{y^{-2}}{-2} + c_1 = \\
&= \frac{2}{y} - \frac{2}{y^2} + c_1 = \frac{2y-2}{y^2} + c_1 = \frac{2(1+t^2)-2}{(1+t^2)^2} + c_1 = \frac{2t^2}{(1+t^2)^2} + c_1 = \frac{2 \operatorname{tg}^2 \frac{x}{2}}{\left(1 + \operatorname{tg}^2 \frac{x}{2}\right)^2} + c_1 = \\
&= \frac{2 \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}{\left(1 + \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}\right)^2} + c_1 = 2 \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \cdot \left( \frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} \right)^2 + c_1 = 2 \sin^2 \frac{x}{2} \cdot \cos^2 \frac{x}{2} + c_1 = \\
&= \frac{1}{2} \left( 4 \sin^2 \frac{x}{2} \cdot \cos^2 \frac{x}{2} \right) + c_1 = \frac{1}{2} \left( 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2} \right)^2 + c_1 = \underline{\underline{\frac{1}{2} \sin^2 x + c_1}}
\end{aligned}$$

## 1.2. Substitute $\operatorname{tg} x = t$

V pravoúhlém trojúhelníku pro úhel  $x$  je:  $\operatorname{tg} x = \frac{t}{1}$

Tím máme stanoveny velikosti obou odvěsen.

Velikost přepony určíme podle Pythagorovy věty.



$$\begin{aligned}
\int \sin x \cos x dx &= \left| \begin{array}{l} t = \operatorname{tg} x \\ dt = \frac{1}{\cos^2 x} dx \\ dx = \frac{1}{1+t^2} dt \end{array} \quad \begin{array}{l} \sin x = \frac{t}{\sqrt{1+t^2}} \\ \cos x = \frac{1}{\sqrt{1+t^2}} \end{array} \right| = \int \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} \cdot \frac{1}{1+t^2} dt = \\
&= \int \frac{t}{(1+t^2)^2} dt = \left| \begin{array}{l} 1+t^2 = z \\ 2t dt = dz \\ dt = \frac{1}{2t} dz \end{array} \right| = \int \frac{t}{z^2} \cdot \frac{1}{2t} dz = \frac{1}{2} \int z^{-2} dz = \frac{1}{2} \cdot \frac{z^{-1}}{-1} + c_2 = \\
&= -\frac{1}{2z} + c_2 = -\frac{1}{2(1+t^2)} + c_2 = -\frac{1}{2(1+\operatorname{tg}^2 x)} + c_2 = -\frac{1}{2\left(1 + \frac{\sin^2 x}{\cos^2 x}\right)} + c_2 = \\
&= -\frac{\cos^2 x}{2(\underbrace{\cos^2 x + \sin^2 x}_1)} + c_2 = \left| c_2 = \frac{1}{2} + c_1 \right| = -\frac{1}{2} \cos^2 x + \frac{1}{2} \cdot 1 + c_1 = \\
&= -\frac{1}{2} \cos^2 x + \frac{1}{2} \cdot (\sin^2 x + \cos^2 x) + c_1 = \underline{\underline{\frac{1}{2} \sin^2 x + c_1}}
\end{aligned}$$

**1.3. Metoda per partes**

$$\int \sin x \cos x \, dx = \left| \begin{array}{ll} u = \sin x & v' = \cos x \\ u' = \cos x & v = \sin x \end{array} \right| = \sin x \sin x - \int \cos x \sin x \, dx$$

$$\int \sin x \cos x \, dx = \sin x \sin x - \int \sin x \cos x \, dx$$

$$2 \int \sin x \cos x \, dx = \sin^2 x$$

$$\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + c_3 = |c_3 = c_1| = \underline{\underline{\frac{1}{2} \sin^2 x + c_1}}$$

**1.4. Substitute**  $\sin x = y$ 

$$\begin{aligned} \int \sin x \cos x \, dx &= \int \sin x \cdot \underline{\cos x \, dx} = \left| \begin{array}{l} \sin x = y \\ \cos x \, dx = dy \end{array} \right| = \int y \, dy = \frac{y^2}{2} + c_4 = \\ &= \frac{\sin^2 x}{2} + c_4 = |c_4 = c_1| = \underline{\underline{\frac{1}{2} \sin^2 x + c_1}} \end{aligned}$$

**1.5. Substitute**  $\cos x = w$ 

$$\begin{aligned} \int \sin x \cos x \, dx &= - \int \cos x \cdot \underline{(-\sin x) \, dx} = \left| \begin{array}{l} \cos x = w \\ -\sin x \, dx = dw \end{array} \right| = - \int w \, dw = \\ &= -\frac{w^2}{2} + c_5 = -\frac{\cos^2 x}{2} + c_5 = \left| c_5 = \frac{1}{2} + c_1 \right| = -\frac{\cos^2 x}{2} + \frac{1}{2} + c_1 = \\ &= -\frac{\cos^2 x}{2} + \frac{\sin^2 x + \cos^2 x}{2} + c_1 = \underline{\underline{\frac{1}{2} \sin^2 x + c_1}} \end{aligned}$$

**1.6. Pomocí vzorce**  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ 

$$\begin{aligned} \int \sin x \cos x \, dx &= \frac{1}{2} \int 2 \sin x \cos x \, dx = \frac{1}{2} \int \sin 2x \, dx = \left| \begin{array}{l} 2x = z \\ 2 \, dx = dz \\ dx = \frac{1}{2} \, dz \end{array} \right| = \\ &= \frac{1}{2} \int \sin z \cdot \frac{1}{2} \, dz = -\frac{1}{4} \cos z + c_6 = -\frac{1}{4} \cos 2x + c_6 = -\frac{1}{4} (\cos^2 x - \sin^2 x) + c_6 = \\ &= \left| c_6 = \frac{1}{4} + c_1 \right| = -\frac{1}{4} (\cos^2 x - \sin^2 x) + \frac{1}{4} (\sin^2 x + \cos^2 x) + c_1 = \underline{\underline{\frac{1}{2} \sin^2 x + c_1}} \end{aligned}$$

## Další příklady na integrování goniometrických funkcí

$$\begin{aligned}
 2. \quad [x \neq (2k+1)\pi] \quad & \int \frac{1 + \sin x}{1 + \cos x} dx = \int \frac{1 + \sin x}{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}\right) + \cos 2\frac{x}{2}} dx = \\
 & = \int \frac{1 + \sin x}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} dx = \int \frac{1 + \sin x}{2 \cos^2 \frac{x}{2}} dx = \\
 & = \frac{1}{2} \int \frac{1}{\cos^2 \frac{x}{2}} dx + \frac{1}{2} \int \frac{\sin 2\frac{x}{2}}{\cos^2 \frac{x}{2}} dx = \left| \begin{array}{l} \frac{x}{2} = w \\ x = 2w \\ dx = 2 dw \end{array} \right| = \\
 & = \frac{1}{2} \int \frac{2}{\cos^2 w} dw + \frac{1}{2} \int \frac{2 \sin 2w}{\cos^2 w} dw = \\
 & = \int \frac{1}{\cos^2 w} dw + \int \frac{2 \sin w \cos w}{\cos^2 w} dw = \operatorname{tg} w + 2 \int \frac{\sin w}{\cos w} dw = \\
 & = \operatorname{tg} \frac{x}{2} - 2 \int \frac{-\sin w}{\cos w} dw = \operatorname{tg} \frac{x}{2} - 2 \int \frac{(\cos w)'}{\cos w} dw = \operatorname{tg} \frac{x}{2} - 2 \ln |\cos w| + c = \\
 & \qquad \qquad \qquad = \underline{\underline{\operatorname{tg} \frac{x}{2} - \ln \left(\cos \frac{x}{2}\right)^2 + c}}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad [x \in \mathbb{R}] \quad & \int \cos^5 x \cdot \sin^2 x dx = \int (\cos^2 x)^2 \cdot \cos x \cdot \sin^2 x dx = \\
 & = \int (1 - \sin^2 x)^2 \cdot \sin^2 x \cdot \underline{\cos x dx} = \left| \begin{array}{l} \sin x = y \\ \cos x dx = dy \end{array} \right| = \int (1 - y^2)^2 y^2 dy = \\
 & = \int (1 - 2y^2 + y^4) \cdot y^2 dy = \int (y^2 - 2y^4 + y^6) dy = \frac{y^3}{3} - 2\frac{y^5}{5} + \frac{y^7}{7} + c = \\
 & \qquad \qquad \qquad = \underline{\underline{\frac{1}{3} \sin^3 x - \frac{2}{5} \sin^5 x + \frac{1}{7} \sin^7 x + c}}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad [x \in \mathbb{R}] \quad & \int \frac{\sin^3 x}{1 + \cos^2 x} dx = \int \frac{(-\sin^2 x) \cdot (-\sin x)}{1 + \cos^2 x} dx = \\
 & = \int \frac{(\cos^2 x - 1)}{1 + \cos^2 x} \underline{(-\sin x) dx} = \left| \begin{array}{l} \cos x = y \\ -\sin x dx = dy \end{array} \right| = \int \frac{y^2 - 1}{y^2 + 1} dy = \\
 & = \int \frac{(y^2 + 1) - 2}{y^2 + 1} dy = \int dy - 2 \int \frac{1}{y^2 + 1} dy = \\
 & \qquad \qquad \qquad = y - 2 \operatorname{arctg} y + c = \underline{\underline{\cos x - 2 \operatorname{arctg}(\cos x) + c}}
 \end{aligned}$$

$$\begin{aligned}
5. \quad [x \neq \arctg \frac{-1}{2} + k\pi]^2 \quad \int \frac{\sin x + 2 \cos x}{\cos x + 2 \sin x} dx &= \left| \begin{array}{l} \operatorname{tg} x = u \\ dx = \frac{1}{1+u^2} du \end{array} \quad \begin{array}{l} \sin x = \frac{u}{\sqrt{u^2+1}} \\ \cos x = \frac{1}{\sqrt{u^2+1}} \end{array} \right|_3 = \\
&= \int \frac{\frac{u}{\sqrt{u^2+1}} + \frac{2}{\sqrt{u^2+1}}}{\frac{1}{\sqrt{u^2+1}} + \frac{2u}{\sqrt{u^2+1}}} \cdot \frac{1}{1+u^2} du = \int \frac{u+2}{1+2u} \cdot \frac{1}{1+u^2} du = \\
&= \left| \begin{array}{l} \frac{u+2}{(1+2u)(1+u^2)} = \frac{A}{1+2u} + \frac{Bu+C}{1+u^2} \\ u+2 = A(1+u^2) + (Bu+C)(1+2u) \\ u = -\frac{1}{2} : \quad \frac{3}{2} = \frac{5}{4}A \quad A = \frac{6}{5} \\ u = 0 : \quad 2 = A+C \quad C = \frac{4}{5} \\ u^2 : \quad 0 = A+2B \quad B = -\frac{3}{5} \end{array} \right| = \\
&= \int \left( \frac{\frac{6}{5}}{1+2u} + \frac{-\frac{3}{5}u + \frac{4}{5}}{1+u^2} \right) du = \\
&= \frac{3}{5} \int \frac{2}{1+2u} du - \frac{3}{10} \int \frac{2u}{1+u^2} du + \frac{4}{5} \int \frac{1}{1+u^2} du = \\
&= \frac{3}{5} \int \frac{(1+2u)'}{1+2u} du - \frac{3}{10} \int \frac{(1+u^2)'}{1+u^2} du + \frac{4}{5} \arctg u = \\
&= \frac{3}{5} \ln(1+2u) - \frac{3}{10} \ln(1+u^2) + \frac{4}{5} \arctg(\operatorname{tg} x) + c = \\
&= \underline{\underline{\frac{3}{10} \ln \frac{(1+2 \operatorname{tg} x)^2}{1+\operatorname{tg}^2 x} + \frac{4}{5} x + c}}
\end{aligned}$$

$$\begin{aligned}
6. \quad [x \in \mathbb{R}] \quad \int \frac{1}{1+3 \cos^2 x} dx &= \quad | \operatorname{tg} x = v |^4 \quad = \int \frac{1}{1+\frac{3}{v^2+1}} \cdot \frac{1}{1+v^2} dv = \\
&= \int \frac{v^2+1}{(v^2+1)+3} \cdot \frac{1}{1+v^2} dv = \int \frac{1}{v^2+4} dv = \left| \begin{array}{l} v = \sqrt{4}w \\ dv = 2 dw \end{array} \right| = \int \frac{1}{4w^2+4} 2 dw = \\
&= \int \frac{2}{4(w^2+1)} dw = \frac{1}{2} \arctg w + c = \frac{1}{2} \arctg \frac{v}{2} + c = \underline{\underline{\frac{1}{2} \arctg \left( \frac{\operatorname{tg} x}{2} \right) + c}}
\end{aligned}$$

<sup>2</sup> Definiční obor:  $\cos x + 2 \sin x \neq 0$   
 $2 \sin x \neq -\cos x$   
 $\frac{\sin x}{\cos x} \neq -\frac{1}{2}$   
 $\operatorname{tg} x \neq -\frac{1}{2}$

<sup>3</sup> Viz příklad 1.2.

<sup>4</sup> Viz příklad 5.

$$\begin{aligned}
7. \quad [x \in \mathbb{R}] \quad & \int \cos^4 x \cdot \sin^3 x \, dx = \int \cos^4 x \cdot (-\sin^2 x) \cdot (-\sin x) \, dx = \\
& = \int \cos^4 x \cdot (\cos^2 x - 1) \cdot \underline{(-\sin x) \, dx} = \left| \begin{array}{l} \cos x = y \\ -\sin x \, dx = dy \end{array} \right| = \int y^4 (y^2 - 1) \, dy = \\
& = \int (y^6 - y^4) \, dy = \frac{y^7}{7} - \frac{y^5}{5} + c = \underline{\underline{\frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + c}}
\end{aligned}$$

$$\begin{aligned}
8. \quad [x \in \mathbb{R}] \quad & \int \cos^2 x \cdot \sin^2 x \, dx = \left| \begin{array}{l} \cos^2 x = \frac{1+\cos 2x}{2} \\ \sin^2 x = \frac{1-\cos 2x}{2} \end{array} \right| = \left| \begin{array}{l} 1 = \sin^2 x + \cos^2 x \\ \cos 2x = \cos^2 x - \sin^2 x \\ 1 + \cos 2x = 2 \cos^2 x \end{array} \right| = \\
& = \int \frac{1 + \cos 2x}{2} \cdot \frac{1 - \cos 2x}{2} \, dx = \int \frac{1 - \cos^2 2x}{4} \, dx = \int \frac{1 - \frac{1+\cos 4x}{2}}{4} \, dx = \\
& = \frac{1}{4} \int \left( \frac{2 - 1 - \cos 4x}{2} \right) \, dx = \frac{1}{8} \int (1 - \cos 4x) \, dx = \left| \begin{array}{l} 4x = v \\ 4 \, dx = dv \\ dx = \frac{1}{4} dv \end{array} \right| = \\
& = \frac{1}{8} x - \frac{1}{8} \int \cos v \cdot \frac{1}{4} \, dv = \frac{1}{8} x - \frac{1}{32} \sin v + c = \underline{\underline{\frac{1}{8} x - \frac{1}{32} \sin 4x + c}}
\end{aligned}$$

$$\begin{aligned}
9. \quad [x \in \mathbb{R}] \quad & \int \sin^3 x \cdot \cos^2 x \, dx = \int -\sin^2 x \cdot (-\sin x) \cdot \cos^2 x \, dx = \\
& = \int -(1 - \cos^2 x) \cdot \cos^2 x \cdot \underline{(-\sin x) \, dx} = \left| \begin{array}{l} \cos x = w \\ -\sin x \, dx = dw \end{array} \right| = \int (w^2 - 1)w^2 \, dw = \\
& = \int w^4 \, dw - \int w^2 \, dw = \frac{w^5}{5} - \frac{w^3}{3} + c = \underline{\underline{\frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + c}}
\end{aligned}$$

$$\begin{aligned}
10. \quad [\text{pro } x \in \langle 0; \frac{\pi}{2} \rangle] \quad & \int \sin^3 x \cdot \sqrt{9 \cos^4 x \sin^2 x + 9 \sin^4 x \cos^2 x} \, dx = \\
& = \int \sin^3 x \cdot \sqrt{9 \sin^2 x \cos^2 x \overbrace{(\cos^2 x + \sin^2 x)}^1} \, dx = \int \sin^3 x \cdot 3 \sin x \cdot \cos x \, dx = \\
& = 3 \int \sin^4 x \cdot \underline{\cos x \, dx} = \left| \begin{array}{l} \sin x = v \\ \cos x \, dx = dv \end{array} \right| = 3 \int v^4 \, dv = 3 \frac{v^5}{5} + c = \underline{\underline{\frac{3}{5} \sin^5 x + c}}
\end{aligned}$$

11.  $\left[ \begin{array}{l} \text{Definičním oborem} \\ \text{se nebudeme zabývat} \end{array} \right]^5 \int \frac{1}{\sin^2 x + 3 \sin x \cos x + 2 \cos^2 x} dx =$

$$= \left| \begin{array}{ll} \operatorname{tg} x = u & \sin x = \frac{u}{\sqrt{u^2+1}} \\ dx = \frac{1}{1+u^2} du & \cos x = \frac{1}{\sqrt{u^2+1}} \end{array} \right| \text{ (viz příklad 1.2.) } =$$

$$= \int \frac{1}{\frac{u^2}{u^2+1} + 3 \frac{u}{\sqrt{u^2+1}} \frac{1}{\sqrt{u^2+1}} + 2 \frac{1}{u^2+1}} \cdot \frac{1}{1+u^2} du = \int \frac{u^2+1}{u^2+3u+2} \cdot \frac{1}{1+u^2} du =$$

$$= \int \frac{1}{(u+1)(u+2)} du = \int \left( \frac{1}{u+1} - \frac{1}{u+2} \right) du =$$

$$= \int \left( \frac{(u+1)'}{u+1} - \frac{(u+2)'}{u+2} \right) du = \ln(u+1) - \ln(u+2) + c = \ln \frac{u+1}{u+2} + c =$$

$$= \ln \frac{\operatorname{tg} x + 1}{\operatorname{tg} x + 2} + c$$

12.  $[x \in \mathbb{R}] \int \frac{2 - \sin x}{2 + \cos x} dx = \int \frac{2}{2 + \cos x} dx + \int \frac{-\sin x}{2 + \cos x} dx =$

$$= \int \frac{2}{2 + \cos x} dx + \int \frac{(2 + \cos x)'}{2 + \cos x} dx = \left| \begin{array}{ll} \operatorname{tg} \frac{x}{2} = t & \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2 dt}{1+t^2} & \cos x = \frac{1-t^2}{1+t^2} \end{array} \right|^6 =$$

$$= \int \frac{2}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt + \ln(2 + \cos x) = \ln(2 + \cos x) + \int \frac{4}{2 + 2t^2 + 1 - t^2} dt =$$

$$= \ln(2 + \cos x) + \int \frac{4}{t^2 + 3} dt = \left| \begin{array}{l} t = \sqrt{3}v \\ dt = \sqrt{3} dv \end{array} \right| = \ln(2 + \cos x) + \int \frac{4}{3v^2 + 3} \cdot \sqrt{3} dv =$$

$$= \ln(2 + \cos x) + \frac{4\sqrt{3}}{3} \int \frac{1}{v^2 + 1} dv = \ln(2 + \cos x) + \frac{4\sqrt{3}}{3} \operatorname{arctg} v + c =$$

$$= \ln(2 + \cos x) + \frac{4\sqrt{3}}{3} \operatorname{arctg} \frac{t}{\sqrt{3}} + c = \ln(2 + \cos x) + \frac{4\sqrt{3}}{3} \operatorname{arctg} \frac{\operatorname{tg} \frac{x}{2}}{\sqrt{3}} + c$$

<sup>5</sup> Kořeny jmenovatele nepatří do definičního oboru. První odhad hodnoty nějakého kořene jmenovatele provedeme grafickou metodou. Ten potom zpřesníme například metodou bisekce (půlení intervalu) a následně zohledníme periodicitu goniometrických funkcí.

<sup>6</sup> Viz příklad 1.1.

13. [Definičním oborem se nebudeme zabývat]<sup>7</sup>  $\int \frac{1 + 2 \sin x}{\sin x + 3 \cos x + 3} dx =$

$$= \left| \begin{array}{l} \operatorname{tg} \frac{x}{2} = t \quad \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2 dt}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \end{array} \right|_8 = \int \frac{1 + 2 \frac{2t}{1+t^2}}{\frac{2t}{1+t^2} + 3 \frac{1-t^2}{1+t^2} + 3} \cdot \frac{2}{1+t^2} dt =$$

$$= \int \frac{\frac{1+t^2+4t}{1+t^2}}{\frac{2t+3-3t^2+3+3t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{t^2+4t+1}{1+t^2} \cdot \frac{1+t^2}{2t+6} \cdot \frac{2}{1+t^2} dt =$$

$$= \int \frac{t^2+4t+1}{2(t+3)} \cdot \frac{2}{1+t^2} dt = \int \left( \frac{A}{t+3} + \frac{Bt+C}{1+t^2} \right) dt =$$

$$t^2 + 4t + 1 = A \cdot (1+t^2) + (Bt+C) \cdot (t+3)$$

$$t = -3 : \quad 9 - 12 + 1 = A \cdot (1+9) \quad A = -\frac{1}{5}$$

$$t = 0 : \quad 1 = A + 3C \quad \xRightarrow{A} \quad C = \frac{2}{5}$$

$$t^2 : \quad 1 = A + B \quad \xRightarrow{A} \quad B = \frac{6}{5}$$

$$= \int \frac{-\frac{1}{5}}{t+3} dt + \int \frac{\frac{6}{5}t + \frac{2}{5}}{1+t^2} dt = -\frac{1}{5} \int \frac{(t+3)'}{t+3} dt + \frac{3}{5} \int \frac{2t}{1+t^2} dt + \frac{2}{5} \int \frac{1}{1+t^2} dt =$$

$$= -\frac{1}{5} \ln |t+3| + \frac{3}{5} \ln(1+t^2) + \frac{2}{5} \operatorname{arctg} t + c = \frac{1}{5} \ln \frac{(1+t^2)^3}{|t+3|} + \frac{2}{5} \operatorname{arctg} t + c =$$

$$= \frac{1}{5} \ln \frac{\left(1 + \operatorname{tg}^2 \frac{x}{2}\right)^3}{\left|3 + \operatorname{tg} \frac{x}{2}\right|} + \frac{2}{5} \operatorname{arctg} \left(\operatorname{tg} \frac{x}{2}\right) + c$$


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14. [ $x \in \mathbb{R}$ ]  $\int \sin^3 x \, dx = \int \sin^2 x \cdot (\sin x) \, dx = \int (1 - \cos^2 x) \cdot (\sin x) \, dx =$

$$= \int (\cos^2 x - 1) \cdot (-\sin x) \, dx = \left| \begin{array}{l} \cos x = w \\ (-\sin x) \, dx = dw \end{array} \right| = \int (w^2 - 1) \, dw =$$

$$= \frac{1}{3} w^3 - w + c = \frac{1}{3} \cos^3 x - \cos x + c$$


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<sup>7</sup> Viz poznámka 5.<sup>8</sup> Viz příklad 1.1.



$$\begin{aligned}
15. \quad [x \neq \frac{k\pi}{2}] \quad \int \frac{1}{\sin^4 x \cdot \cos^4 x} dx &= \left| \begin{array}{ll} \operatorname{tg} x = u & \sin x = \frac{u}{\sqrt{u^2+1}} \\ dx = \frac{1}{u^2+1} du & \cos x = \frac{1}{\sqrt{u^2+1}} \end{array} \right|_9 = \\
&= \int \frac{1}{\left(\frac{u}{\sqrt{u^2+1}}\right)^4 \cdot \left(\frac{1}{\sqrt{u^2+1}}\right)^4} \cdot \frac{1}{u^2+1} du = \int \frac{(u^2+1)^{\frac{4}{2}} \cdot (u^2+1)^{\frac{4}{2}}}{u^4 \cdot 1^4 \cdot (u^2+1)} du = \\
&= \int \frac{(u^2+1)^3}{u^4} du = \int \frac{u^6 + 3u^4 + 3u^2 + 1}{u^4} du = \int (u^2 + 3 + 3u^{-2} + u^{-4}) du = \\
&= \frac{1}{3}u^3 + 3u + 3\frac{u^{-1}}{-1} + \frac{u^{-3}}{-3} + c = \underline{\underline{\frac{1}{3}\operatorname{tg}^3 x + 3\operatorname{tg} x - \frac{3}{\operatorname{tg} x} - \frac{1}{3\operatorname{tg}^3 x} + c}}
\end{aligned}$$

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<sup>9</sup> Viz příklad 1.2.