

$$\begin{aligned}
 \text{1. a) } [x \in (-2; 2)] \quad \int \frac{x}{\sqrt{4-x^2}} dx &= \left| \begin{array}{l} 4-x^2 = s \\ -2x dx = ds \\ dx = \frac{-1}{2x} ds \end{array} \right| = \int \frac{x}{\sqrt{s}} \cdot \frac{-1}{2x} ds = \\
 &= -\frac{1}{2} \int s^{-\frac{1}{2}} ds = -\frac{1}{2} \cdot \frac{s^{\frac{1}{2}}}{\frac{1}{2}} + c = -\sqrt{s} + c = \underline{\underline{-\sqrt{4-x^2} + c}}
 \end{aligned}$$

$$\begin{aligned}
 \text{1. b) } [x \in (-2; 2)] \quad \int \frac{x}{\sqrt{4-x^2}} dx &= \left| \begin{array}{l} \sqrt{4-x^2} = t \\ 4-x^2 = t^2 \\ -2x dx = 2t dt \\ dx = \frac{-t}{x} dt \end{array} \right| = \int \frac{x}{t} \cdot \frac{-t}{x} dt = \\
 &= -\int dt = -t + c = \underline{\underline{-\sqrt{4-x^2} + c}}
 \end{aligned}$$

$$\begin{aligned}
 \text{2. a) } [x \in \mathbb{R}] \quad \int \frac{2x-1}{\sqrt{1-x+x^2}} dx &= \left| \begin{array}{l} 1-x+x^2 = u \\ (-1+2x) dx = du \\ dx = \frac{1}{2x-1} du \end{array} \right| = \\
 &= \int \frac{2x-1}{\sqrt{u}} \cdot \frac{1}{2x-1} du = \int u^{-\frac{1}{2}} du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{2 \cdot \sqrt{1-x+x^2} + c}}
 \end{aligned}$$

$$\begin{aligned}
 \text{2. b) } [x \in \mathbb{R}] \quad \int \frac{2x-1}{\sqrt{1-x+x^2}} dx &= \left| \begin{array}{l} \sqrt{1-x+x^2} = v \\ 1-x+x^2 = v^2 \\ (-1+2x) dx = 2v dv \\ dx = \frac{2v}{2x-1} dv \end{array} \right| = \\
 &= \int \frac{2x-1}{v} \cdot \frac{2v}{2x-1} dv = 2 \int dv = 2v + c = \underline{\underline{2 \cdot \sqrt{1-x+x^2} + c}}
 \end{aligned}$$

$$\begin{aligned}
3. \quad [x \in (0; 1) \cup (1; \infty)] \quad & \int \frac{\sqrt{x}}{\sqrt[3]{x^2} - \sqrt[4]{x}} dx = \int \frac{x^{\frac{1}{2}}}{x^{\frac{2}{3}} - x^{\frac{1}{4}}} dx = \\
& = \left[ \frac{1}{2}; \frac{2}{3}; \frac{1}{4} \Rightarrow \text{nejmenší společný jmenovatel je } 12 \Rightarrow \begin{array}{l} \sqrt[12]{x} = z \\ x = z^{12} \\ dx = 12z^{11} dz \end{array} \right] = \\
& = \int \frac{(z^{12})^{\frac{1}{2}}}{(z^{12})^{\frac{2}{3}} - (z^{12})^{\frac{1}{4}}} 12z^{11} dz = 12 \int \frac{z^6}{z^8 - z^3} z^{11} dz = 12 \int \frac{z^{17}}{z^3(z^5 - 1)} dz = \\
& = 12 \int \frac{z^{14}}{z^5 - 1} dz = 12 \int \frac{z^{10}}{z^5 - 1} z^4 dz = \left[ \begin{array}{l} z^5 - 1 = t \\ z^5 = t + 1 \\ 5z^4 dz = dt \\ z^4 dz = \frac{1}{5} dt \\ z^{10} = (z^5)^2 \end{array} \right] = \\
& = 12 \int \frac{(t+1)^2}{t} \cdot \frac{1}{5} dt = \frac{12}{5} \int \frac{t^2 + 2t + 1}{t} dt = \frac{12}{5} \int \left( t + 2 + \frac{1}{t} \right) dt = \\
& = \frac{12}{5} \left( \frac{t^2}{2} + 2t + \ln |t| \right) + c = \frac{12}{10} [(z^5 - 1)^2 + 4(z^5 - 1) + 2 \ln |(z^5 - 1)|] + c = \\
& = \frac{12}{10} \left[ \left( \sqrt[12]{x^5} - 1 \right)^2 + 4 \left( \sqrt[12]{x^5} - 1 \right) + \ln \left( \sqrt[12]{x^5} - 1 \right)^2 \right] + c
\end{aligned}$$

$$\begin{aligned}
4. \quad [x \in (-1; 5)] \quad & \int \frac{1}{\sqrt{4x + 5 - x^2}} dx = \int \frac{1}{\sqrt{5 - (x^2 - 4x)}} dx = \\
& = \int \frac{1}{\sqrt{5 - (x^2 - 4x + 4 - 4)}} dx = \int \frac{1}{\sqrt{5 - (x^2 - 4x + 4) + 4}} dx = \\
& = \int \frac{1}{\sqrt{9 - (x - 2)^2}} dx = \int \frac{1}{\sqrt{9 \left[ 1 - \frac{(x-2)^2}{9} \right]}} dx = \int \frac{1}{\sqrt{9} \cdot \sqrt{1 - \left( \frac{x-2}{3} \right)^2}} dx = \\
& = \frac{1}{3} \int \frac{1}{\sqrt{1 - \left( \frac{x-2}{3} \right)^2}} dx = \left[ \begin{array}{l} \frac{x-2}{3} = y \\ x = 3y + 2 \\ dx = 3 dy \end{array} \right] = \frac{1}{3} \int \frac{1}{\sqrt{1 - y^2}} \cdot 3 dy = \arcsin y + c = \\
& \quad \quad \quad = \arcsin \frac{x-2}{3} + c
\end{aligned}$$

$$\begin{aligned}
5. \quad [x \in (-2; 2)] \quad \int \frac{x^2}{\sqrt{4-x^2}} dx &= \left| \begin{array}{l} x = 2 \sin y \\ dx = 2 \cos y \, dy \\ \frac{x}{2} = \sin y \\ \arcsin \frac{x}{2} = y \end{array} \right| = \\
&= \int \frac{(2 \sin y)^2}{\sqrt{4 - (2 \sin y)^2}} \cdot 2 \cos y \, dy = \int \frac{4 \sin^2 y}{\sqrt{4 - 4 \sin^2 y}} \cdot 2 \cos y \, dy = \\
&= \int \frac{8 \sin^2 y \cdot \cos y}{\sqrt{4} \cdot \sqrt{1 - \sin^2 y}} \, dy = \int \frac{8 \sin^2 y \cdot \cos y}{2 \cdot \sqrt{\cos^2 y}} \, dy = \\
&= 4 \int \frac{\sin^2 y \cdot \cos y}{|\cos y|} \, dy = \left| \begin{array}{l} \arcsin \frac{x}{2} = y \\ H\left(\arcsin \frac{x}{2}\right) = \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \\ \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) = D(y) \\ \cos y > 0 \end{array} \right| = 4 \int \frac{\sin^2 y \cdot \cos y}{\cos y} \, dy = \\
&= 4 \int \sin^2 y \, dy = 4 \int \frac{1 - \cos 2y}{2} \, dy = 2 \int dy - \int \cos 2y \cdot \underline{2 \, dy} = \left| \begin{array}{l} 2y = z \\ \underline{2 \, dy} = dz \end{array} \right| = \\
&= 2y - \int \cos z \, dz = 2 \arcsin \frac{x}{2} - \sin z + c = 2 \arcsin \frac{x}{2} - \sin 2y + c = \\
&= 2 \arcsin \frac{x}{2} - 2 \sin y \cdot \cos y + c = 2 \arcsin \frac{x}{2} - 2 \cdot \frac{x}{2} \cdot \sqrt{1 - \sin^2 y} + c = \\
&= 2 \arcsin \frac{x}{2} - x \cdot \sqrt{1 - \left(\frac{x}{2}\right)^2} + c = 2 \arcsin \frac{x}{2} - x \cdot \sqrt{\frac{4 - x^2}{4}} + c = \\
&= \underline{\underline{2 \arcsin \frac{x}{2} - \frac{x}{2} \cdot \sqrt{4 - x^2} + c}}
\end{aligned}$$

6. a)  $[x \in (-\infty; -2) \cup (2; \infty)]$   $\int \sqrt{x^2 - 4} \, dx =$

$$\begin{aligned}
 &= \left| \begin{array}{l} \sqrt{x^2 - 4} = v - x \\ x^2 - 4 = v^2 - 2vx + x^2 \\ 2vx = v^2 + 4 \\ x = \frac{v^2 + 4}{2v} \\ dx = \frac{2v(2v) - (v^2 + 4)2}{(2v)^2} \, dv \\ dx = \frac{v^2 - 4}{2v^2} \, dv \\ x + \sqrt{x^2 - 4} = v \end{array} \right| = \int \left( v - \frac{v^2 + 4}{2v} \right) \cdot \frac{v^2 - 4}{2v^2} \, dv = \\
 &= \int \frac{[2v^2 - (v^2 + 4)] \cdot (v^2 - 4)}{2v \cdot 2v^2} \, dv = \int \frac{(v^2 - 4)^2}{4v^3} \, dv = \int \frac{v^4 - 8v^2 + 16}{4v^3} \, dv = \\
 &= \int \frac{v^4}{4v^3} \, dv - \frac{8}{4} \int \frac{v^2}{v^3} \, dv + \int \frac{16}{4v^3} \, dv = \frac{1}{4} \int v \, dv - 2 \int \frac{1}{v} \, dv + 4 \int v^{-3} \, dv = \\
 &= \frac{1}{4} \cdot \frac{v^2}{2} - 2 \ln |v| + 4 \cdot \frac{v^{-2}}{-2} + c = \frac{v^2}{8} - \ln v^2 - \frac{2}{v^2} + c = \\
 &= \frac{v^4 - 16}{8v^2} - \ln v^2 + c = \frac{(x + \sqrt{x^2 - 4})^4 - 16}{8(\sqrt{x^2 - 4} + x)^2} - \ln(x + \sqrt{x^2 - 4})^2 + c = \dots = \\
 &= \underline{\underline{\frac{x}{2} \cdot \sqrt{x^2 - 4} - \ln(x + \sqrt{x^2 - 4})^2 + c}}^1
 \end{aligned}$$

<sup>1</sup> Tento výsledek dává webová stránka

[https://www.wolframalpha.com/input/?i=integrate+sqrt\(x^2-4\)](https://www.wolframalpha.com/input/?i=integrate+sqrt(x^2-4))

$$6. b) \quad [x \in (-\infty; -2) \cup (2; \infty)] \quad \int \sqrt{x^2 - 4} \, dx = \left| \begin{array}{l} x = \frac{2}{\sin w} = 2 \sin^{-1} w \\ dx = -2 \sin^{-2} w \cdot \cos w \, dw \\ dx = \frac{-2 \cos w}{\sin^2 w} \, dw \\ \sin w = \frac{2}{x} \\ w = \arcsin \frac{2}{x} \Rightarrow w \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \end{array} \right|$$

$$= \int \sqrt{\left(\frac{2}{\sin w}\right)^2 - 4} \cdot \frac{-2 \cos w}{\sin^2 w} \, dw = \int \sqrt{\frac{4(1 - \sin^2 w)}{\sin^2 w}} \cdot \frac{-2 \cos w}{\sin^2 w} \, dw =$$

$$= \int \frac{\sqrt{4} \cdot \sqrt{1 - \sin^2 w}}{\sqrt{\sin^2 w}} \cdot \frac{-2 \cos w}{\sin^2 w} \, dw = \left| \begin{array}{l} \sqrt{1 - \sin^2 w} = z \\ 1 - \sin^2 w = z^2 \\ \sin^2 w = 1 - z^2 \\ \sin w = \sqrt{1 - z^2} \\ \sqrt{\cos^2 w} = z \\ |\cos w| = z \\ w \in \dots : \cos w = z \\ -\sin w \, dw = dz \\ dw = \frac{-1}{\sqrt{1 - z^2}} \, dz \end{array} \right| =$$

$$= \int \frac{2z}{\sqrt{1 - z^2}} \cdot \frac{-2z}{1 - z^2} \cdot \frac{-1}{\sqrt{1 - z^2}} \, dz = \int \frac{4z^2}{(1 - z^2)^2} \, dz = \int \frac{4z^2}{(z^2 - 1)^2} \, dz =$$

$$= \int \frac{4z^2}{[(z + 1)(z - 1)]^2} \, dz = \int \left[ \frac{A}{z + 1} + \frac{B}{(z + 1)^2} + \frac{C}{z - 1} + \frac{D}{(z - 1)^2} \right] \, dz =$$

$$\left| \begin{array}{l} 4z^2 = A(z + 1)(z - 1)^2 + B(z - 1)^2 + C(z + 1)^2(z - 1) + D(z + 1)^2 \\ z = 1 : 4 = 4D \quad D = 1 \\ z = -1 : 4 = 4B \quad B = 1 \\ z^3 : 0 = A + C \quad A = -C \\ z = 0 : 0 = A + B - C + D \\ \quad \quad 0 = A + 1 + A + 1 \quad A = -1 \\ \quad \quad \quad \quad \quad C = 1 \end{array} \right|$$

$$= \int \left[ \frac{-1}{z + 1} + \frac{1}{(z + 1)^2} + \frac{1}{z - 1} + \frac{1}{(z - 1)^2} \right] \, dz =$$

$$\begin{aligned}
&= - \int \frac{(z+1)'}{z+1} dz + \int \frac{1}{(z+1)^2} dz + \int \frac{(z-1)'}{z-1} dz + \int \frac{1}{(z-1)^2} dz = \\
&= \left| \begin{array}{l} z+1=u \\ dz=du \end{array} \right| \left| \begin{array}{l} z-1=v \\ dz=dv \end{array} \right| = -\ln|z+1| + \int \frac{1}{u^2} du + \ln|z-1| + \int \frac{1}{v^2} dv = \\
&= \ln \left| \frac{z-1}{z+1} \right| + \int u^{-2} du + \int v^{-2} dv = \ln \left| \frac{z-1}{z+1} \right| + \frac{u^{-1}}{-1} + \frac{v^{-1}}{-1} + c = \\
&= \ln \left| \frac{z-1}{z+1} \right| - \frac{1}{u} - \frac{1}{v} + c = \ln \left| \frac{z-1}{z+1} \cdot \frac{z-1}{z-1} \right| - \frac{1}{z+1} - \frac{1}{z-1} + c = \\
&= \ln \left| \frac{z^2 - 2z + 1}{z^2 - 1} \right| - \frac{(z-1) + (z+1)}{z^2 - 1} + c = \ln \left| \frac{z^2 - 2z + 1}{z^2 - 1} \right| - \frac{2z}{z^2 - 1} + c = \\
&= \ln \left| \frac{1 - \sin^2 w - 2 \cdot \sqrt{1 - \sin^2 w} + 1}{1 - \sin^2 w - 1} \right| - \frac{2 \cdot \sqrt{1 - \sin^2 w}}{1 - \sin^2 w - 1} + c = \\
&= \ln \left| \frac{1 - \left(\frac{2}{x}\right)^2 - 2 \cdot \sqrt{1 - \left(\frac{2}{x}\right)^2} + 1}{1 - \left(\frac{2}{x}\right)^2 - 1} \right| - \frac{2 \cdot \sqrt{1 - \left(\frac{2}{x}\right)^2}}{1 - \left(\frac{2}{x}\right)^2 - 1} + c = \\
&= \ln \left| \left( 1 - \frac{4}{x^2} - 2 \cdot \sqrt{1 - \frac{4}{x^2}} + 1 \right) \cdot \frac{x^2}{-4} \right| - \left( 2 \cdot \sqrt{1 - \frac{4}{x^2}} \right) \cdot \frac{x^2}{-4} + c = \\
&= \underline{\underline{\frac{x}{2} \cdot \sqrt{x^2 - 4} - \ln \left( x + \sqrt{x^2 - 4} \right)^2 + c}}
\end{aligned}$$

**6. c)**  $[x \in (-\infty; -2) \cup (2; \infty)]$   $\int \sqrt{x^2 - 4} dx = \left| \begin{array}{l} x = e^y + e^{-y} \\ dx = (e^y - e^{-y}) dy \end{array} \right| =$

$$\begin{aligned}
&= \int \sqrt{(e^y + e^{-y})^2 - 4} \cdot (e^y - e^{-y}) dy = \\
&= \int \sqrt{e^{2y} + 2 \underbrace{e^{2y} e^{-2y}}_{e^0=1} + e^{-2y} - 4} \cdot (e^y - e^{-y}) dy =
\end{aligned}$$

$$= \int \sqrt{(e^y - e^{-y})^2} \cdot (e^y - e^{-y}) dy = \underbrace{\int |e^y - e^{-y}| \cdot (e^y - e^{-y}) dy}_{J} =$$

$$\mathcal{A} : y > 0 \quad \Rightarrow \quad J = \int (e^y - e^{-y}) \cdot (e^y - e^{-y}) dy = \int (e^y - e^{-y})^2 dy =$$

$$= \int (e^{2y} - 2 + e^{-2y}) dy = \frac{1}{2} \cdot e^{2y} - 2y + \frac{1}{-2} \cdot e^{-2y} + c_1 =$$

Jestliže:  $x = e^y + e^{-y}$  pak:  $y = \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)$ , protože:

$$x = e^{\ln \frac{x + \sqrt{x^2 - 4}}{2}} + e^{-\ln \frac{x + \sqrt{x^2 - 4}}{2}} = e^{\ln \frac{x + \sqrt{x^2 - 4}}{2}} + e^{\ln\left(\frac{x + \sqrt{x^2 - 4}}{2}\right)^{-1}} =$$

$$= \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2}{x + \sqrt{x^2 - 4}} = \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2}{x + \sqrt{x^2 - 4}} \cdot \frac{x - \sqrt{x^2 - 4}}{x - \sqrt{x^2 - 4}} =$$

$$= \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2 \cdot (x - \sqrt{x^2 - 4})}{x^2 - (x^2 - 4)} = \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2 \cdot (x - \sqrt{x^2 - 4})}{4} = x$$

$$= \frac{1}{2} \cdot e^{2 \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)} - 2 \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right) - \frac{1}{2} \cdot e^{-2 \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)} + c_1 =$$

$$= \frac{1}{2} \cdot \left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)^2 - \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)^2 - \frac{1}{2 \cdot \left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)^2} + c_1 =$$

$$= \frac{1}{8} \cdot (x + \sqrt{x^2 - 4})^2 - \frac{2}{(x + \sqrt{x^2 - 4})^2} \cdot \frac{(x - \sqrt{x^2 - 4})^2}{(x - \sqrt{x^2 - 4})^2} -$$

$$- \ln\left[\left(x + \sqrt{x^2 - 4}\right) \cdot \frac{1}{2}\right]^2 + c_1 =$$

$$= \frac{1}{8} \cdot (x + \sqrt{x^2 - 4})^2 - \frac{2 \cdot (x^2 - \sqrt{x^2 - 4})^2}{[x^2 - (x^2 - 4)]^2} - \ln(x + \sqrt{x^2 - 4})^2 + \underbrace{\ln 4 + c_1}_c =$$

$$= \frac{1}{8} \cdot (x + \sqrt{x^2 - 4})^2 - \frac{(x^2 - \sqrt{x^2 - 4})^2}{8} - \ln(x + \sqrt{x^2 - 4})^2 + c =$$

$$= \frac{x^2 + 2x\sqrt{x^2 - 4} + (x^2 - 4) - [x^2 - 2x\sqrt{x^2 - 4} + (x^2 - 4)]}{8} - \ln(x + \sqrt{x^2 - 4})^2 + c =$$

$$= \underline{\underline{\frac{x}{2} \cdot \sqrt{x^2 - 4} - \ln(x + \sqrt{x^2 - 4})^2 + c}}$$

$$\begin{aligned}
\mathcal{B} : y > 0 &\Rightarrow \mathcal{J} = \int -(e^y - e^{-y}) \cdot (e^y - e^{-y}) dy = \\
&= - \int (e^{2y} - 2 + e^{-2y}) dy = - \int (e^y - e^{-y})^2 dy = - \int (e^{2y} - 2 + e^{-2y}) dy = \\
&= -\frac{1}{2} \cdot e^{2y} + 2y - \frac{1}{-2} \cdot e^{-2y} + c_2 = \\
&= -\frac{1}{2} \cdot e^{2 \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)} + 2 \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right) + \frac{1}{2} \cdot e^{-2 \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)} + c_2 = \\
&= -\frac{1}{2} \left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)^2 + \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)^2 + \frac{1}{2} \left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)^{-2} + c_2 = \\
&= \dots = \underline{\underline{\frac{x}{2} \cdot \sqrt{x^2 - 4} - \ln\left(x + \sqrt{x^2 - 4}\right)^2 + c}}
\end{aligned}$$

$$\begin{aligned}
\text{6. d) } [x \in (-\infty; -2) \cup (2; \infty)] \quad &\underline{\int \sqrt{x^2 - 4} \, dx} = \int \sqrt{(x+2) \cdot (x-2)} \, dx = \\
&= \int \underbrace{\sqrt{(x+2)^2 \cdot \frac{x-2}{x+2}}}_{\text{pro } x \neq -2} dx = \left[ \begin{array}{l} \sqrt{\frac{x-2}{x+2}} = u, \quad \text{kde } u \in (0; 1), \text{ pro } x \in (2; \infty) \\ \frac{x-2}{x+2} = u^2 \quad \text{a } u \in (1; \infty), \text{ pro } x \in (-\infty; -2) \\ x - 2 = u^2(x + 2) \\ x - u^2x = 2u^2 + 2 \\ x(1 - u^2) = 2u^2 + 2 \\ x = \frac{2u^2 + 2}{1 - u^2} \\ dx = \frac{4u(1-u^2) - (2u^2+2)(-2u)}{(1-u^2)^2} du \\ dx = \frac{8u}{(1-u^2)^2} du \end{array} \right] \\
&= \int \sqrt{\left(\frac{2u^2 + 2}{1 - u^2} + 2\right)^2} \cdot u^2 \cdot \frac{8u}{(1 - u^2)^2} du = \int \sqrt{\left(\frac{4}{1 - u^2}\right)^2} \cdot \sqrt{u^2} \cdot \frac{8u}{(1 - u^2)^2} du = \\
&= \int \frac{4}{|1 - u^2|} \cdot u \cdot \frac{8u}{(1 - u^2)^2} du = \underbrace{\int \frac{32u^2}{|1 - u^2| \cdot (1 - u^2)^2} du}_j =
\end{aligned}$$



$$\begin{aligned}\mathcal{A} : u \in \langle 0; 1) \quad \Rightarrow \quad \mathcal{I} &= \int \frac{32u^2}{(1-u^2) \cdot (1-u^2)^2} du = \int \frac{32u^2}{(1-u^2)^3} du = \\ &= \int \frac{32u^2}{[(1+u)(1-u)]^3} du \\ \frac{32u^2}{[(1+u)(1-u)]^3} &= \frac{A}{1+u} + \frac{B}{(1+u)^2} + \frac{C}{(1+u)^3} + \frac{D}{1-u} + \frac{E}{(1-u)^2} + \frac{F}{(1-u)^3} \\ 32u^2 &= A(1+u)^2(1-u)^3 + B(1+u)(1-u)^3 + C(1-u)^3 + D(1+u)^3(1-u)^2 + E(1+u)^3(1-u) + F(1+u)^3 \\ u = -1 : 32 &= 8C \quad \Rightarrow \quad C = 4 \\ u = 1 : 32 &= 8F \quad \Rightarrow \quad F = 4\end{aligned}$$

$$32u^2 = A(1+2u+u^2)(1-u)^3 + B(1+u)(1-u)^3 + C(1-3u+3u^2-u^3) + D(1+u)^3(1-2u+u^2) + E(1+u)^3(1-u) + F(1+3u+3u^2+u^3)$$

$$\begin{aligned}u^5 : 0 &= -5A + 5D & A &= D \\ u^2 : 32 &= A(3-6+1) + B(3-3) + C(3) + D(3-6+1) + E(-3+3) + F(3) \\ 32 &= -2D + 4 \cdot 3 - 2D + 4 \cdot 3 & A, D &= -2 \\ u^4 : 0 &= A(3-2) + B(-1) + D(3-2) + E(-1) \\ 0 &= -2 \cdot 1 - B - 2 \cdot 1 - E & B &= -4 - E \\ u^1 : 0 &= A(2-3) + B(1-3) + C(-3) + D(3-2) + E(3-1) + F(3) \\ 0 &= -2 \cdot (-1) + (-4-E) \cdot (-2) + 4 \cdot (-3) - 2 \cdot 1 + 2E + 4 \cdot 3 & B, E &= -2\end{aligned}$$

$$\begin{aligned}&= \int \left[ \frac{-2}{1+u} + \frac{-2}{(1+u)^2} + \frac{4}{(1+u)^3} + \frac{-2}{1-u} + \frac{-2}{(1-u)^2} + \frac{4}{(1-u)^3} \right] du = \\ &= \left| \begin{array}{l} 1+u=v \\ du=dv \end{array} \right| \left| \begin{array}{l} 1-u=w \\ -du=dw \end{array} \right| = \\ &= -2 \int \frac{1}{v} dv - 2 \int v^{-2} dv + 4 \int v^{-3} dv + 2 \int \frac{1}{w} dw + 2 \int w^{-2} dw - 4 \int w^{-3} dw = \\ &= -2 \ln |v| - 2 \frac{v^{-1}}{-1} + 4 \frac{v^{-2}}{-2} + 2 \ln |w| + 2 \frac{w^{-1}}{-1} - 4 \frac{w^{-2}}{-2} + c = \\ &= \frac{2}{v} - \frac{2}{v^2} - \frac{2}{w} + \frac{2}{w^2} - \ln v^2 + \ln w^2 + c = \frac{2v-2}{v^2} + \frac{-2w+2}{w^2} - (\ln v^2 - \ln w^2) + c = \\ &= \frac{2v-2}{v^2} - \frac{2w-2}{w^2} - \ln \frac{v^2}{w^2} + c = \frac{2(1+u)-2}{(1+u)^2} - \frac{2(1-u)-2}{(1-u)^2} - \ln \frac{(1+u)^2}{(1-u)^2} + c = \\ &= \frac{2u}{(1+u)^2} + \frac{2u}{(1-u)^2} - \ln \left( \frac{1+u}{1-u} \right)^2 + c = \\ &= \frac{2 \cdot \sqrt{\frac{x-2}{x+2}}}{\left( 1 + \sqrt{\frac{x-2}{x+2}} \right)^2} + \frac{2 \cdot \sqrt{\frac{x-2}{x+2}}}{\left( 1 - \sqrt{\frac{x-2}{x+2}} \right)^2} - \ln \left( \frac{1 + \sqrt{\frac{x-2}{x+2}}}{1 - \sqrt{\frac{x-2}{x+2}}} \right)^2 + c = \dots = \\ &= \underline{\underline{\frac{x}{2} \cdot \sqrt{x^2-4} - \ln \left( 2x + 2 \cdot \sqrt{x^2-4} \right)^2 + c}}\end{aligned}$$

$$\mathcal{B} : u \in (1; \infty) \quad \Rightarrow \quad \mathcal{I} = \int \dots$$

<sup>2</sup> Tento výsledek dává webová stránka

[http://um.mendelu.cz/maw-html/index.php?lang=cs&form=integral&function=sqrt\(x^2-4\)](http://um.mendelu.cz/maw-html/index.php?lang=cs&form=integral&function=sqrt(x^2-4))

$$\begin{aligned}
7. \quad [x \in \langle -2; -1 \rangle \cup (-1; \infty)] \quad \int \frac{x+1}{x+\sqrt{x+2}} dx &= \left| \begin{array}{l} \sqrt{x+2} = v \\ x+2 = v^2 \\ x = v^2 - 2 \\ dx = 2v dv \end{array} \right| = \\
&= \int \frac{(v^2 - 2) + 1}{(v^2 - 2) + v} \cdot 2v dv = \int \frac{v^2 - 1}{v^2 + v - 2} \cdot 2v dv = \int \frac{(v+1)(v-1) \cdot 2v}{(v+2)(v-1)} dv = \\
&= \int \frac{(v+1) \cdot 2v}{v+2} dv = \int \frac{[(v+2) - 1] \cdot 2v}{v+2} dv = \\
&= \int \frac{(v+2) \cdot 2v}{v+2} dv + \int \frac{[-1] \cdot 2v}{v+2} dv = 2 \int v dv + \int \frac{-2[(v+2) - 2]}{v+2} dv = \\
&= 2 \cdot \frac{v^2}{2} + \int \frac{-2(v+2)}{v+2} dv + \int \frac{-2[-2]}{v+2} dv = v^2 - 2 \int dv + 4 \int \frac{(v+2)'}{v+2} dv = \\
&= v^2 - 2v + 4 \ln |v+2| + c = \underline{\underline{x+2 - 2 \cdot \sqrt{x+2} + \ln(2 + \sqrt{x+2})^4 + c}}
\end{aligned}$$

$$\begin{aligned}
8. \quad [x \in \langle 0; \infty \rangle] \quad \int \frac{\sqrt{x}}{1 + \sqrt[4]{x^3}} dx &= \left| \begin{array}{l} \sqrt[4]{x} = y \quad (\text{viz př. 3}) \\ x = y^4 \\ dx = 4y^3 dy \\ x^{\frac{1}{6}} = y \end{array} \right| = \int \frac{\sqrt{y^4}}{1 + \sqrt[4]{(y^4)^3}} \cdot 4y^3 dy = \\
&= \int \frac{y^2}{1 + y^3} \cdot 4y^3 dy = \int \frac{4y^5}{y^3 + 1} dy = \int \frac{4y^2[(y^3 + 1) - 1]}{y^3 + 1} dy = \\
&= \int \frac{4y^2(y^3 + 1)}{y^3 + 1} dy + \int \frac{4y^2[-1]}{y^3 + 1} dy = 4 \int y^2 dy - \frac{4}{3} \int \frac{3y^2}{y^3 + 1} dy = \\
&= \left| \begin{array}{l} y^3 = z \\ 3y^2 dy = dz \\ dy = \frac{dz}{3y^2} \end{array} \right| = 4 \frac{y^3}{3} - \frac{4}{3} \int \frac{3y^2}{z+1} \cdot \frac{1}{3y^2} dz = 4 \frac{y^3}{3} - \frac{4}{3} \int \frac{(z+1)'}{z+1} dz = \\
&= 4 \frac{y^3}{3} - \frac{4}{3} \ln |z+1| + c = \frac{4}{3} y^3 - \frac{1}{3} \ln (y^3 + 1)^4 + c = \\
&= \underline{\underline{\frac{4}{3} \cdot \sqrt[4]{x^3} - \frac{1}{3} \ln (\sqrt[4]{x^3} + 1)^4 + c}}
\end{aligned}$$

$$\begin{aligned}
9. \quad & \left[ x \in \mathbb{R} \setminus \left\{ -\frac{5}{3} \right\} \right] \quad \int \frac{\sqrt[3]{3x+4}}{1 + \sqrt[3]{3x+4}} dx = \left| \begin{array}{l} \sqrt[3]{3x+4} = u \\ 3x+4 = u^3 \\ 3 dx = 3u^2 du \\ dx = u^2 du \end{array} \right| = \int \frac{u}{1+u} \cdot u^2 du = \\
& = \int \frac{u^2[(u+1)-1]}{u+1} du = \int \frac{u^2(u+1)}{u+1} du + \int \frac{u^2[-1]}{u+1} du = \\
& = \int u^2 du + \int \frac{(-u)u}{u+1} du = \int u^2 du + \int \frac{(-u)[(u+1)-1]}{u+1} du = \\
& = \frac{u^3}{3} + \int \frac{(-u)(u+1)}{u+1} du + \int \frac{(-u)[-1]}{u+1} du = \frac{1}{3}u^3 - \int u du + \int \frac{u}{u+1} du = \\
& = \frac{1}{3}u^3 - \frac{u^2}{2} + \int \frac{(u+1)-1}{u+1} du = \frac{1}{3}u^3 - \frac{1}{2}u^2 + \int \frac{(u+1)}{u+1} du + \int \frac{-1}{u+1} du = \\
& = \frac{1}{3}u^3 - \frac{1}{2}u^2 + \int du - \int \frac{(u+1)'}{u+1} du = \frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1| + c = \\
& = \underline{\underline{\frac{1}{3}(3x+4) - \frac{1}{2} \cdot \sqrt[3]{(3x+4)^2} + \sqrt[3]{3x+4} - \ln|1 + \sqrt[3]{3x+4}| + c}}
\end{aligned}$$

$$\begin{aligned}
10. \quad & \left[ x \in \mathbb{R} \setminus \{1 \pm \sqrt{6}\} \right] \quad \int \frac{8x-11}{\sqrt{5+2x-x^2}} dx = \int \frac{-4(-2x + \frac{11}{4})}{\sqrt{5+2x-x^2}} dx = \\
& = \int \frac{-4[(2-2x)-2 + \frac{11}{4}]}{\sqrt{5+2x-x^2}} dx = \int \frac{-4(2-2x)}{\sqrt{5+2x-x^2}} dx + \int \frac{-4[\frac{3}{4}]}{\sqrt{5+2x-x^2}} dx = \\
& = \left| \begin{array}{l} \sqrt{5+2x-x^2} = v \\ 5+2x-x^2 = v^2 \\ (2-2x) dx = 2v dv \\ dx = \frac{2v}{2-2x} dv \end{array} \right| = \int \frac{-4(2-2x)}{v} \cdot \frac{2v}{2-2x} dv + \int \frac{-3}{\sqrt{5+2x-x^2}} dx = \\
& = \int -8 dv + \int \frac{-3}{\sqrt{5-(x^2-2x)}} dx = -8 \int dv - 3 \int \frac{1}{\sqrt{5-(x^2-2x+1-1)}} dx =
\end{aligned}$$

$$\begin{aligned}
&= -8v - 3 \int \frac{1}{\sqrt{5 - (x^2 - 2x + 1) + 1}} dx = -8v - 3 \int \frac{1}{\sqrt{6 - (x - 1)^2}} dx = \\
&= -8v - 3 \int \frac{1}{\sqrt{6 \left[ 1 - \frac{(x-1)^2}{6} \right]}} dx = -8v - 3 \int \frac{1}{\sqrt{6} \cdot \sqrt{1 - \left( \frac{x-1}{\sqrt{6}} \right)^2}} dx = \\
&= \left| \begin{array}{l} \frac{x-1}{\sqrt{6}} = w \\ x-1 = w\sqrt{6} \\ dx = \sqrt{6} dw \end{array} \right| = -8v - 3 \int \frac{\sqrt{6}}{\sqrt{6} \cdot \sqrt{1-w^2}} dw = -8v - 3 \arcsin w + c = \\
&= \underline{\underline{-8\sqrt{5+2x-x^2} - 3 \arcsin \left( \frac{x-1}{\sqrt{6}} \right) + c}}
\end{aligned}$$

**11. a)**  $[x \in (-\infty; -1) \cup (0; \infty)]$   $\int \frac{1}{x + \sqrt{x^2 + x}} dx = \int \frac{1}{x + \sqrt{x(x+1)}} dx =$

$$\begin{aligned}
&= \int \frac{1}{x + \sqrt{x^2 \cdot \frac{x+1}{x}}} dx = \int \frac{1}{x + \sqrt{x^2} \cdot \sqrt{\frac{x+1}{x}}} dx = \left| \begin{array}{l} \sqrt{\frac{x+1}{x}} = z \\ \frac{x+1}{x} = z^2 \\ x+1 = x \cdot z^2 \\ 1 = x \cdot z^2 - 1 \\ x = \frac{1}{z^2 - 1} \\ dx = \frac{-2z}{(z^2 - 1)^2} dz \end{array} \right| = \\
&= \int \frac{1}{\frac{1}{z^2-1} + \sqrt{\left(\frac{1}{z^2-1}\right)^2} \cdot z} \cdot \frac{-2z}{(z^2-1)^2} dz = \\
&= \int \frac{1}{\left[ \frac{1}{z^2-1} + z \cdot \sqrt{\left(\frac{1}{z^2-1}\right)^2} \right] \cdot (z^2-1)} \cdot \frac{-2z}{z^2-1} dz = \\
&= \int \frac{1}{1 + z \cdot \sqrt{\frac{1}{(z^2-1)^2}} \cdot (z^2-1)^2} \cdot \frac{-2z}{z^2-1} dz = \int \frac{1}{1 + z\sqrt{1}} \cdot \frac{-2z}{(z+1)(z-1)} dz =
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{-2z}{(z+1)^2(z-1)} dz = \int \left[ \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-1} \right] dz = \\
&= \left| \begin{array}{l} -2z = A(z+1)(z-1) + B(z-1) + C(z+1)^2 \\ z=1: \quad -2 = 4C \\ z=-1: \quad 2 = -2B \\ z=0: \quad 0 = -A - B + C \end{array} \right. \begin{array}{l} C = \frac{-1}{2} \\ B = -1 \\ A = \frac{1}{2} \end{array} \left. \right| = \\
&= \int \frac{\frac{1}{2}}{z+1} dz + \underbrace{\int \frac{-1}{(z+1)^2} dz}_{\mathcal{I}} + \int \frac{\frac{-1}{2}}{z-1} dz = \\
&= \frac{1}{2} \int \frac{(z+1)'}{z+1} dz + \mathcal{I} - \frac{1}{2} \int \frac{(z-1)'}{z-1} dz =
\end{aligned}$$

$$\left| \begin{array}{l} \text{Výpočet integrálu } \mathcal{I} \quad (\text{nebo substitute } |z+1=t|) \\ \int \frac{-1}{z+1} dz = \left| \begin{array}{ll} u = \frac{-1}{z+1} & v' = 1 \\ u' = \frac{1}{(z+1)^2} & v = z \end{array} \right| = \frac{-1}{z+1} \cdot z - \int \frac{1}{(z+1)^2} \cdot z dz = \frac{-z}{z+1} - \int \frac{(z+1)-1}{(z+1)^2} dz \\ \int \frac{-1}{z+1} dz = \frac{-z}{z+1} - \int \frac{1}{z+1} dz - \mathcal{I} \\ \mathcal{I} = \frac{-z}{z+1} + c_1 \end{array} \right|$$

$$\begin{aligned}
&= \frac{1}{2} \ln|z+1| + \frac{-z}{z+1} - \frac{1}{2} \ln|z-1| + c_a = \frac{-z}{z+1} + \frac{1}{2} \ln \left| \frac{z+1}{z-1} \right| + c_a = \\
&= \frac{-z}{z+1} \cdot \frac{z-1}{z-1} + \frac{1}{2} \ln \left| \frac{z+1}{z-1} \cdot \frac{z+1}{z+1} \right| + c_a = \frac{-z^2+z}{z^2-1} + \frac{1}{2} \ln \left| \frac{(z+1)^2}{z^2-1} \right| + c_a = \\
&= \frac{-z^2+z}{z^2-1} + \frac{1}{2} \ln \left| \frac{z^2+2z+1}{z^2-1} \right| + c_a = \\
&= \frac{-\left(\sqrt{\frac{x+1}{x}}\right)^2 + \sqrt{\frac{x+1}{x}}}{\left(\sqrt{\frac{x+1}{x}}\right)^2 - 1} + \frac{1}{2} \ln \left| \frac{\left(\sqrt{\frac{x+1}{x}}\right)^2 + 2 \cdot \sqrt{\frac{x+1}{x}} + 1}{\left(\sqrt{\frac{x+1}{x}}\right)^2 - 1} \right| + c_a = \\
&= \frac{-\frac{x+1}{x} + \sqrt{\frac{x+1}{x}}}{\frac{x+1}{x} - 1} + \frac{1}{2} \ln \left| \frac{\frac{x+1}{x} + 2 \cdot \sqrt{\frac{x+1}{x}} + 1}{\frac{x+1}{x} - 1} \right| + c_a =
\end{aligned}$$

$$\begin{aligned}
&= \frac{-\frac{x+1}{x} + \frac{1}{x} \cdot \sqrt{x(x+1)}}{\frac{x+1-x}{x}} + \frac{1}{2} \ln \left| \frac{\frac{x+1+x}{x} + \frac{2}{x} \cdot \sqrt{x(x+1)}}{\frac{x+1-x}{x}} \right| + c_a = \\
&= \left[ \frac{-x-1+\sqrt{x^2+x}}{x} \right] \cdot \frac{x}{1} + \frac{2}{4} \ln \left| \left[ \frac{2x+1+2\sqrt{x^2+x}}{x} \right] \cdot \frac{x}{1} \right| + c_a = \\
&= -x \underbrace{-1 + c_a}_c + \sqrt{x^2+x} + \frac{1}{4} \ln (2x+1+2\sqrt{x^2+x})^2 = \\
&= \underline{\underline{-x + \sqrt{x^2+x} + \frac{1}{4} \ln (2x+1+2\sqrt{x^2+x})^2 + c}}
\end{aligned}$$

**11. b)**  $[x \in (-\infty; -1) \cup (0; \infty)]$

$$\int \frac{1}{x + \sqrt{x^2+x}} dx = \left| \begin{array}{l} \sqrt{x^2+x} = t-x \\ x^2+x = t^2-2tx+x^2 \\ 2tx+x = t^2 \\ x = \frac{t^2}{2t+1} \\ dx = \frac{2t(2t+1)-2t^2}{(1+2t)^2} dt \\ dx = \frac{2t^2+2t}{(2t+1)^2} dt \end{array} \right| =$$

$$= \int \frac{1}{x + (t-x)} \cdot \frac{2t^2+2t}{(2t+1)^2} dt = \int \frac{t(2t+2)}{t(2t+1)^2} dt = \int \left[ \frac{A}{2t+1} + \frac{B}{(2t+1)^2} \right] dt =$$

$$\left| \begin{array}{l} 2t+2 = A(2t+1) + B \\ t = \frac{-1}{2} : \quad 1 = B \quad B = 1 \\ t^1 : \quad 2 = 2A \quad A = 1 \end{array} \right|$$

$$= \int \frac{1}{2t+1} dt + \underbrace{\int \frac{1}{(2t+1)^2} dt}_J = \frac{1}{2} \int \frac{2}{2t+1} dt + J = \frac{1}{2} \int \frac{(2t+1)'}{2t+1} dt + J =$$

<p>Výpočet integrálu <math>\mathcal{J}</math> (nebo substituce <math>  2t + 1 = w  </math>)</p> $\int \frac{1}{2t+1} dt = \left  \begin{array}{ll} u = \frac{1}{2t+1} & v' = 1 \\ u' = \frac{-2}{(2t+1)^2} & v = t \end{array} \right  = \frac{1}{2t+1} \cdot t - \int \frac{-2}{(2t+1)^2} \cdot t dt = \frac{t}{2t+1} + \int \frac{(2t+1)-1}{(2t+1)^2} dt$ $\int \frac{1}{2t+1} dt = \frac{t}{2t+1} + \int \frac{1}{2t+1} dt - \mathcal{J}$ $\mathcal{J} = \frac{t}{2t+1} + c_2$
-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

$$= \frac{1}{2} \ln |2t + 1| + \frac{t}{2t+1} + c = \frac{t}{2t+1} + \frac{1}{4} \ln(2t + 1)^2 + c =$$

$$= \frac{x + \sqrt{x^2 + x}}{2(x + \sqrt{x^2 + x}) + 1} + \frac{1}{4} \ln [2(x + \sqrt{x^2 + x}) + 1]^2 + c =$$

$$= \frac{x + \sqrt{x^2 + x}}{(2x + 1) + 2\sqrt{x^2 + x}} \cdot \frac{(2x + 1) - 2\sqrt{x^2 + x}}{(2x + 1) - 2\sqrt{x^2 + x}} + \frac{1}{4} \ln (2x + 1 + 2\sqrt{x^2 + x})^2 + c =$$

$$= \frac{(x + \sqrt{x^2 + x}) \cdot (2x + 1 - 2\sqrt{x^2 + x})}{(2x + 1)^2 - 4(x^2 + x)} + \frac{1}{4} \ln (2x + 1 + 2\sqrt{x^2 + x})^2 + c =$$

$$= \frac{2x^2 + x - 2x\sqrt{x^2 + x} + 2x\sqrt{x^2 + x} + \sqrt{x^2 + x} - 2(x^2 + x)}{\textcolor{red}{4x^2} + \textcolor{green}{4x} + 1 - \textcolor{red}{4x^2} - \textcolor{green}{4x}} + \frac{1}{4} \ln (2x + 1 + 2\sqrt{x^2 + x})^2 + c =$$

$$= \frac{2x^2 + x + \sqrt{x^2 + x} - 2x^2 - 2x}{1} + \frac{1}{4} \ln (2x + 1 + 2\sqrt{x^2 + x})^2 + c =$$

$$= \underline{\underline{-x + \sqrt{x^2 + x} + \frac{1}{4} \ln (2x + 1 + 2\sqrt{x^2 + x})^2 + c}}$$

12.  $[x \in (-\infty; -2) \cup (1; \infty)] \quad \int \frac{1}{\sqrt{x^2 + x - 2}} dx =$

$$\begin{aligned}
 &= \left| \begin{array}{l} \sqrt{x^2 + x - 2} = w - x \\ x^2 + x - 2 = w^2 - 2wx + x^2 \\ 2wx + x = w^2 + 2 \\ x = \frac{w^2 + 2}{2w + 1} \\ dx = \frac{2w(2w+1) - (w^2+2)2}{(2w+1)^2} dw \\ dx = \frac{2w^2 + 2w - 4}{(2w + 1)^2} dw \end{array} \right| = \int \frac{1}{w - \frac{w^2+2}{2w+1}} \cdot \frac{2w^2 + 2w - 4}{(2w + 1)^2} dw = \\
 &= \int \frac{1}{\frac{2w^2 + w - (w^2+2)}{2w+1}} \cdot \frac{2w^2 + 2w - 4}{(2w + 1)^2} dw = \\
 &= \int \frac{2w + 1}{w^2 + w - 2} \cdot \frac{2(w^2 + w - 2)}{(2w + 1)^2} dw = \int \frac{2}{2w + 1} dw = \int \frac{(2w + 1)'}{2w + 1} dw = \\
 &= \ln |2w + 1| + c = \ln \left| 2 \left( x + \sqrt{x^2 + x - 2} \right) + 1 \right| + c = \\
 &= \underline{\underline{\frac{1}{2} \ln \left( 2x + 1 + 2\sqrt{x^2 + x - 2} \right)^2 + c}}
 \end{aligned}$$

13.  $[x \in \mathbb{R}]$

$$\int \frac{x}{\sqrt{x^2 + 4x + 5}} dx =$$

$$\begin{aligned}
 &\left| \begin{array}{l} \sqrt{(x+2)^2 + 1} = z - x \\ x^2 + 4x + 5 = z^2 - 2xz + x^2 \\ 2xz + 4x = z^2 - 5 \\ x(2z + 4) = z^2 - 5 \\ x = \frac{z^2 - 5}{2(z + 2)} = \frac{z^2 - 5}{2z + 4} \\ dx = \frac{2z[2(z+2)] - (z^2-5) \cdot 2}{[2(z+2)]^2} dz \\ dx = \frac{2(z^2 + 4z + 5)}{4(z+2)^2} dz \\ dx = \frac{z^2 + 4z + 5}{2(z + 2)^2} dz \end{array} \right| =
 \end{aligned}$$



$$\begin{aligned}
&= \int \frac{\frac{z^2-5}{2z+4}}{z - \frac{z^2-5}{2z+4}} \cdot \frac{z^2+4z+5}{2(z+2)^2} dz = \int \frac{z^2-5}{2z+4} \cdot \frac{2z+4}{2z^2+4z-z^2+5} \cdot \frac{z^2+4z+5}{2(z+2)^2} dz = \\
&= \int \frac{\frac{1}{2}(z^2-5)}{(z+2)^2} dz = \int \frac{\frac{1}{2} \cdot [(z^2+4z+4) - 4z - 9]}{z^2+4z+4} dz = \\
&= \frac{1}{2} \int \frac{(z^2+4z+4)}{z^2+4z+4} dz + \int \frac{\frac{1}{2} \cdot [-4z-9]}{(z+2)^2} dz = \frac{1}{2} \int dz + \int \frac{-2z - \frac{9}{2}}{(z+2)^2} dz = \\
&= \frac{1}{2} \cdot z + \int \frac{-[2(z+2) + \frac{1}{2}]}{(z+2)^2} dz = \frac{1}{2} \cdot z + \int \frac{-2(z+2)}{(z+2)^2} dz + \int \frac{-[\frac{1}{2}]}{(z+2)^2} dz = \\
&= \frac{1}{2} \cdot z - 2 \int \frac{(z+2)'}{z+2} dz - \frac{1}{2} \int \frac{1}{(z+2)^2} dz = \left| \begin{array}{l} z+2 = y \\ dz = dy \end{array} \right| = \\
&= \frac{1}{2} \cdot z - 2 \ln |z+2| = \\
&= \underline{\underline{\sqrt{x^2+4x+5} - \ln \left( x+2 + \sqrt{x^2+4x+5} \right)^2 + c}}
\end{aligned}$$

$$\begin{aligned}
14. \quad [x \in (0; \infty)] \quad \int \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx &= \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx = \left| \begin{array}{l} \sqrt[6]{x} = z \quad (\text{viz př. 3}) \\ x = z^6 \\ dx = 6z^5 dz \\ x^{\frac{1}{6}} = z \end{array} \right| = \\
&= \int \frac{1}{(z^6)^{\frac{1}{2}} + (z^6)^{\frac{1}{3}}} \cdot 6z^5 dz = 6 \int \frac{z^5}{z^3 + z^2} dz = 6 \int \frac{z^2 \cdot \{[z^3+1] - 1\}}{z^2 \cdot (z+1)} dz = \\
&= 6 \int \frac{[(z+1)(z^2-z+1)]}{z+1} dz - 6 \int \frac{\{1\}}{z+1} dz = \\
&= 6 \int (z^2 - z + 1) dz - 6 \int \frac{(z+1)'}{z+1} dz = 6 \left( \frac{z^3}{3} - \frac{z^2}{2} + z - \ln |z+1| \right) + c = \\
&= 2z^3 - 3z^2 + 6z - \ln(z+1)^6 + c = 2 \left( x^{\frac{1}{6}} \right)^3 - 3 \left( x^{\frac{1}{6}} \right)^2 + 6x^{\frac{1}{6}} - \ln \left( 1 + x^{\frac{1}{6}} \right)^6 + c = \\
&= \underline{\underline{2 \cdot \sqrt{x} - 3 \cdot \sqrt[3]{x} + 6 \cdot \sqrt[6]{x} - \ln \left( 1 + \sqrt[6]{x} \right)^6 + c}}
\end{aligned}$$

$$\begin{aligned}
15. \quad [x \in \mathbb{R} \setminus \{-1\}] \quad \int \frac{\sqrt[3]{3x+4}}{x + \sqrt[3]{3x+4}} dx &= \left| \begin{array}{l} \sqrt[3]{3x+4} = u \\ 3x+4 = u^3 \\ x = \frac{u^3-4}{3} \\ dx = u^2 du \end{array} \right| = \\
&= \int \frac{u}{\frac{u^3-4}{3} + u} \cdot u^2 du = \int \frac{3u^3}{u^3 + 3u - 4} du = \int \frac{3[(u^3 + 3u - 4) - 3u + 4]}{u^3 + 3u - 4} du = \\
&= \int \left\{ \frac{3(u^3 + 3u - 4)}{u^3 + 3u - 4} + \frac{3[-3u + 4]}{(u-1)(u^2 + u + 4)} \right\} du = \\
&= 3 \int du + \int \frac{-9u + 12}{(u-1)(u^2 + u + 4)} du = 3 \int du + \int \left( \frac{A}{u-1} + \frac{Bu + C}{u^2 + u + 4} \right) du = \\
&\left| \begin{array}{ll} -9u + 12 = A(u^2 + u + 4) + (Bu + C)(u-1) & \\ u = 1 : & 3 = 6A + 0 \quad A = \frac{1}{2} \\ u = 0 : & 12 = 4A - C \quad C = -10 \\ u^2 : & 0 = A + B \quad B = -\frac{1}{2} \end{array} \right| \\
&= 3u + \int \frac{\frac{1}{2}}{u-1} du + \int \frac{-\frac{1}{2}u - 10}{u^2 + u + 4} du = \\
&= 3u + \frac{1}{2} \int \frac{(u-1)'}{u-1} du + \int \frac{\frac{-1}{4}[(2u+1)-1] - 10}{u^2 + u + 4} du = \\
&= 3u + \frac{1}{2} \ln|u-1| - \frac{1}{4} \int \frac{2u+1}{u^2 + u + 4} du + \int \frac{\frac{1}{4} - 10}{u^2 + u + 4} du = \\
&= 3u + \frac{1}{4} \ln(u-1)^2 - \frac{1}{4} \int \frac{(u^2 + u + 4)'}{u^2 + u + 4} du + \int \frac{-\frac{39}{4}}{u^2 + u + \frac{1}{4} + \frac{15}{4}} du = \\
&= 3u + \frac{1}{4} \ln(u-1)^2 - \frac{1}{4} \ln(u^2 + u + 4) - \frac{39}{4} \int \frac{1}{\left(u + \frac{1}{2}\right)^2 + \frac{15}{4}} du = \\
&= 3u + \frac{1}{4} \ln \frac{(u-1)^2}{u^2 + u + 4} - \frac{39}{4} \int \frac{1}{\frac{15}{4} \left[ \left(\frac{2u+1}{2}\right)^2 \cdot \frac{4}{15} + 1 \right]} du =
\end{aligned}$$

$$\begin{aligned}
&= 3u + \frac{1}{4} \ln \frac{(u-1)^2}{u^2+u+4} - \frac{39}{15} \int \frac{1}{\left(\frac{2u+1}{\sqrt{15}}\right)^2 + 1} du = \left| \begin{array}{l} \frac{2u+1}{\sqrt{15}} = v \\ 2u+1 = v\sqrt{15} \\ u = \frac{v\sqrt{15}-1}{2} \\ du = \frac{\sqrt{15}}{2} dv \end{array} \right| = \\
&= 3u + \frac{1}{4} \ln \frac{(u-1)^2}{u^2+u+4} - \frac{39}{15} \int \frac{1}{v^2+1} \cdot \frac{\sqrt{15}}{2} dv = \\
&= 3u + \frac{1}{4} \ln \frac{(u-1)^2}{u^2+u+4} - \frac{39}{15} \cdot \frac{\sqrt{15}}{2} \operatorname{arctg} v + c = \\
&= 3u + \frac{1}{4} \ln \frac{(u-1)^2}{u^2+u+4} - \frac{39}{2\sqrt{15}} \operatorname{arctg} \frac{2u+1}{\sqrt{15}} + c = \\
&= 3 \cdot \sqrt[3]{3x+4} + \frac{1}{4} \ln \frac{(\sqrt[3]{3x+4}-1)^2}{\sqrt[3]{(3x+4)^2} + \sqrt[3]{3x+4} + 4} - \frac{39}{2 \cdot \sqrt{15}} \operatorname{arctg} \frac{2(\sqrt[3]{3x+4})+1}{\sqrt{15}} + c
\end{aligned}$$


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16.  $[x \in (-1; 0) \cup (0; \infty)]$   $\int \frac{\sqrt{x+1}}{x} dx = \left| \begin{array}{l} \sqrt{x+1} = v \\ x+1 = v^2 \\ x = v^2 - 1 \\ dx = 2v dv \end{array} \right| = \int \frac{v}{v^2-1} \cdot 2v dv =$

$$\begin{aligned}
&= \int \frac{2v^2}{v^2-1} dv = \int \frac{2[(v^2-1)+1]}{v^2-1} dv = \int \frac{2(v^2-1)}{v^2-1} dv + \int \frac{2[1]}{v^2-1} dv = \\
&= 2 \int dv + \int \frac{2}{(v+1)(v-1)} dv = 2v + \int \left( \frac{1}{v-1} - \frac{1}{v+1} \right) dv = \\
&= 2v + \int \left( \frac{1}{v-1} - \frac{1}{v+1} \right) dv = 2v + \int \left( \frac{(v-1)'}{v-1} - \frac{(v+1)'}{v+1} \right) dv = \\
&= 2v + \ln|v-1| - \ln|v+1| + c = 2v + \ln \left| \frac{v-1}{v+1} \right| + c = \\
&= 2\sqrt{x+1} + \ln \left| \frac{-1+\sqrt{x+1}}{1+\sqrt{x+1}} \right| + c
\end{aligned}$$


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$$\begin{aligned}
 17. \quad [x \in (0; \infty)] \quad \int \frac{1}{\sqrt{x} \cdot (1 + \sqrt[3]{x})} dx &= \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{5}{6}}} dx = \left. \begin{array}{l} \sqrt[6]{x} = z \quad (\text{viz př. 3}) \\ x = z^6 \\ dx = 6z^5 dz \\ x^{\frac{1}{6}} = z \end{array} \right| = \\
 &= \int \frac{1}{(z^6)^{\frac{1}{2}} + (z^6)^{\frac{5}{6}}} \cdot 6z^5 dz = \int \frac{6z^5}{z^3 + z^5} dz = \int \frac{6z^3 \cdot [(z^2 + 1) - 1]}{z^3 \cdot (1 + z^2)} dz = \\
 &= \int \frac{6(z^2 + 1)}{1 + z^2} dz - \int \frac{6[1]}{1 + z^2} dz = 6 \int dz - 6 \int \frac{1}{1 + z^2} dz = 6z - 6 \operatorname{arctg} z + c = \\
 &= 6x^{\frac{1}{6}} - 6 \operatorname{arctg} x^{\frac{1}{6}} + c = \underline{\underline{6 \cdot \sqrt[6]{x} - 6 \operatorname{arctg} \sqrt[6]{x} + c}}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad [x \in (0; \infty)] \quad \int \frac{x-1}{\sqrt{x}} dx &= \int \frac{x}{\sqrt{x}} dx + \int \frac{-1}{\sqrt{x}} dx = \int x^{\frac{1}{2}} dx - \int x^{-\frac{1}{2}} dx = \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{\frac{2}{3} \cdot \sqrt{x^3} - 2 \cdot \sqrt{x} + c}}
 \end{aligned}$$

$$\begin{aligned}
 19. \text{ a) } [x \in \mathbb{R}] \quad \int \sqrt{x^2 + 1} dx &= \left. \begin{array}{l} u = (x^2 + 1)^{\frac{1}{2}} \quad v' = 1 \\ u' = \frac{2x}{2 \cdot \sqrt{x^2 + 1}} \quad v = x \end{array} \right| = \\
 &= x \cdot \sqrt{x^2 + 1} - \int \frac{x^2}{\sqrt{x^2 + 1}} dx = x \cdot \sqrt{x^2 + 1} - \int \frac{(x^2 + 1) - 1}{\sqrt{x^2 + 1}} dx = \\
 &= x \cdot \sqrt{x^2 + 1} - \left[ \int \frac{(x^2 + 1)}{\sqrt{x^2 + 1}} dx + \int \frac{-1}{\sqrt{x^2 + 1}} dx \right] \\
 \Rightarrow \quad \int \sqrt{x^2 + 1} dx &= x \cdot \sqrt{x^2 + 1} - \int \sqrt{x^2 + 1} dx + \underbrace{\int \frac{1}{\sqrt{x^2 + 1}} dx}_J \\
 2 \int \sqrt{x^2 + 1} dx &= x \cdot \sqrt{x^2 + 1} + J \\
 \int \sqrt{x^2 + 1} dx &= \frac{x}{2} \cdot \sqrt{x^2 + 1} + \frac{1}{2} \cdot J \\
 \int \sqrt{x^2 + 1} dx &= \underline{\underline{\frac{x}{2} \cdot \sqrt{x^2 + 1} + \frac{1}{2} \cdot \ln(x + \sqrt{x^2 + 1}) + c}}
 \end{aligned}$$

**Výpočet integrálu** 
$$J = \int \frac{1}{\sqrt{x^2 + 1}} dx = \left| \begin{array}{l} \sqrt{x^2 + 1} = w - x \\ x^2 + 1 = w^2 - 2xw + x^2 \\ 2xw = w^2 - 1 \\ x = \frac{w^2 - 1}{2w} \\ dx = \frac{2w \cdot 2w - (w^2 - 1) \cdot 2}{(2w)^2} dw \\ dx = \frac{w^2 + 1}{2w^2} dw \end{array} \right|$$

$$= \int \frac{1}{w - \frac{w^2 - 1}{2w}} \cdot \frac{w^2 + 1}{2w^2} dw = \int \frac{1}{\frac{2w^2 - (w^2 - 1)}{2w}} \cdot \frac{w^2 + 1}{2w^2} dw =$$

$$= \int \frac{2w}{2w^2 - w^2 + 1} \cdot \frac{w^2 + 1}{2w^2} dw = \int \frac{1}{w} dw = \ln |w| + c = \ln \left| x + \underbrace{\sqrt{x^2 + 1}}_{>|x|} \right| + c =$$

$$\underbrace{\hspace{10em}}_{>0} = \ln \left( x + \sqrt{x^2 + 1} \right) + c$$

**19. b)**  $[x \in \mathbb{R}]$  
$$\int \sqrt{x^2 + 1} dx = \left| \begin{array}{l} \sqrt{x^2 + 1} = u - x \\ x^2 + 1 = u^2 - 2xu + x^2 \\ 2xu = u^2 - 1 \\ x = \frac{u^2 - 1}{2u} \\ dx = \frac{2u \cdot 2u - (u^2 - 1) \cdot 2}{(2u)^2} du \\ dx = \frac{2(u^2 + 1)}{2 \cdot 2u^2} du \end{array} \right| =$$

$$= \int \left( u - \frac{u^2 - 1}{2u} \right) \cdot \frac{u^2 + 1}{2u^2} du = \int \frac{2u^2 - (u^2 - 1)}{2u} \cdot \frac{u^2 + 1}{2u^2} du =$$

$$= \int \frac{u^2 + 1}{2u} \cdot \frac{u^2 + 1}{2u^2} du = \int \frac{u^4 + 2u^2 + 1}{4u^3} du = \int \left( \frac{u}{4} + \frac{1}{2u} + \frac{1}{4u^3} \right) du =$$

$$= \frac{1}{4} \int u du + \frac{1}{2} \int \frac{1}{u} du + \frac{1}{4} \int u^{-3} du = \frac{1}{4} \cdot \frac{u^2}{2} + \frac{1}{2} \cdot \ln |u| + \frac{1}{4} \cdot \frac{u^{-2}}{-2} + c =$$

$$= \frac{u^2}{8} + \frac{1}{2} \cdot \ln |u| - \frac{1}{8u^2} + c =$$

$$\begin{aligned}
&= \frac{(x + \sqrt{x^2 + 1})^2}{8} + \frac{1}{2} \cdot \underbrace{\ln |x + \sqrt{x^2 + 1}|}_{>|x|} - \frac{1}{8(x + \sqrt{x^2 + 1})^2} + c = \\
&= \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + \frac{(x + \sqrt{x^2 + 1})^2}{8} - \frac{1}{8(x + \sqrt{x^2 + 1})^2} \cdot \frac{(x - \sqrt{x^2 + 1})^2}{(x - \sqrt{x^2 + 1})^2} + c = \\
&= \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + \frac{(x + \sqrt{x^2 + 1})^2}{8} - \frac{(x - \sqrt{x^2 + 1})^2}{8[(x + \sqrt{x^2 + 1}) \cdot (x - \sqrt{x^2 + 1})]^2} + c = \\
&= \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + \frac{(x + \sqrt{x^2 + 1})^2}{8} - \frac{(x - \sqrt{x^2 + 1})^2}{8[x^2 - (x^2 + 1)]^2} + c = \\
&= \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + \frac{(x + \sqrt{x^2 + 1})^2 - (x - \sqrt{x^2 + 1})^2}{8} + c = \\
&= \frac{\ln(x + \sqrt{x^2 + 1})}{2} + \frac{x^2 + 2x\sqrt{x^2 + 1} + (x^2 + 1) - [x^2 - 2x\sqrt{x^2 + 1} + (x^2 + 1)]}{8} + c = \\
&= \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + \frac{4x\sqrt{x^2 + 1}}{8} + c = \underline{\underline{\frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + \frac{x}{2} \cdot \sqrt{x^2 + 1} + c}}
\end{aligned}$$

19. c) [ $x \in \mathbb{R}$ ]  $\int \sqrt{x^2 + 1} \, dx = \left| \begin{array}{l} x = \operatorname{tg} t \\ dx = \frac{1}{\cos^2 t} \, dt \\ \operatorname{arctg} x = t \\ t \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \end{array} \right. \quad \left. \begin{array}{l} \sin t = \frac{x}{\sqrt{x^2 + 1}} \\ \text{Diagram: A right triangle with horizontal base 1, vertical side } x, \text{ and hypotenuse } \sqrt{x^2 + 1}. \text{ The angle at the origin is } t. \end{array} \right|$

$$\begin{aligned}
&= \int \sqrt{\operatorname{tg}^2 t + 1} \cdot \frac{1}{\cos^2 t} \, dt = \int \sqrt{\frac{\sin^2 t}{\cos^2 t} + 1} \cdot \frac{1}{\cos^2 t} \, dt = \\
&= \int \sqrt{\frac{\sin^2 t + \cos^2 t}{\cos^2 t}} \cdot \frac{1}{\cos^2 t} \, dt = \int \sqrt{\frac{1}{\cos^2 t}} \cdot \frac{1}{\cos^2 t} \, dt = \int \frac{1}{|\cos t|} \cdot \frac{1}{\cos^2 t} \, dt =
\end{aligned}$$

$$\begin{aligned}
&= \left| \frac{t \in \dots}{\cos t > 0} \right| = \int \frac{1}{\cos t} \cdot \frac{1}{\cos^2 t} \cdot \frac{\cos t}{\cos t} dt = \int \frac{\cos t}{(\cos^2 t)^2} dt = \int \frac{\cos t}{(1 - \sin^2 t)^2} dt = \\
&= \left| \begin{array}{l} \sin t = w \\ \cos t dt = dw \\ dt = \frac{1}{\cos t} dw \end{array} \right| = \int \frac{\cos t}{(1 - w^2)^2} \cdot \frac{1}{\cos t} dw = \int \frac{1}{(1 - w^2)^2} dw = \\
&= \int \frac{1}{(w^2 - 1)^2} dw = \int \frac{1}{[(w + 1)(w - 1)]^2} dw = \\
&= \int \frac{A}{w + 1} dw + \int \frac{B}{(w + 1)^2} dw + \int \frac{C}{w - 1} dw + \int \frac{D}{(w - 1)^2} dw = \\
&\left| \begin{array}{l} 1 = A(w + 1)(w - 1)^2 + B(w - 1)^2 + C(w + 1)^2(w - 1) + D(w + 1)^2 \\ w = 1 : 1 = 4D \quad D = \frac{1}{4} \\ w = -1 : 1 = 4B \quad B = \frac{1}{4} \\ w^3 : 0 = A + C \quad A = -C \\ w = 0 : 0 = A + B - C + D \\ 0 = A + \frac{1}{4} + A + \frac{1}{4} \quad A = \frac{1}{4} \\ \quad \quad \quad \quad \quad \quad \quad C = \frac{-1}{4} \end{array} \right| \\
&= \int \frac{\frac{1}{4}}{w + 1} dw + \int \frac{\frac{1}{4}}{(w + 1)^2} dw + \int \frac{\frac{-1}{4}}{w - 1} dw + \int \frac{\frac{1}{4}}{(w - 1)^2} dw = \\
&= \frac{1}{4} \int \frac{(w + 1)'}{w + 1} dw + \frac{1}{4} \int (w + 1)^{-2} dw - \frac{1}{4} \int \frac{(w - 1)'}{w - 1} dw + \frac{1}{4} \int (w - 1)^{-2} dw = \\
&= \frac{1}{4} \ln |w + 1| + \frac{1}{4} \cdot \frac{(w + 1)^{-1}}{-1} - \frac{1}{4} \ln |w - 1| + \frac{1}{4} \cdot \frac{(w - 1)^{-1}}{-1} + c = \\
&= \frac{1}{4} \ln \left| \frac{w + 1}{w - 1} \right| - \frac{1}{4(w + 1)} - \frac{1}{4(w - 1)} + c = \frac{1}{4} \ln \left| \frac{w + 1}{w - 1} \right| - \frac{(w - 1) + (w + 1)}{4(w + 1)(w - 1)} + c = \\
&= \frac{1}{4} \ln \left| \frac{w + 1}{w - 1} \right| - \frac{2w}{4(w^2 - 1)} + c = \frac{1}{4} \ln \left| \frac{w + 1}{w - 1} \right| - \frac{w}{2(w^2 - 1)} + c = \\
&= \frac{1}{4} \ln \left| \frac{\sin t + 1}{\sin t - 1} \right| - \frac{\sin t}{2(\sin^2 t - 1)} + c = \frac{1}{4} \ln \left| \frac{\frac{x}{\sqrt{x^2 + 1}} + 1}{\frac{x}{\sqrt{x^2 + 1}} - 1} \right| - \frac{\frac{x}{\sqrt{x^2 + 1}}}{2 \left[ \left( \frac{x}{\sqrt{x^2 + 1}} \right)^2 - 1 \right]} + c =
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \ln \left| \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} \cdot \frac{\sqrt{x^2 + 1}}{x - \sqrt{x^2 + 1}} \right| - \frac{\frac{x}{\sqrt{x^2 + 1}}}{2 \left[ \frac{x^2}{x^2 + 1} - 1 \right]} + c = \\
&= \frac{1}{4} \ln \left| \frac{x + \sqrt{x^2 + 1}}{x - \sqrt{x^2 + 1}} \right| - \frac{\frac{x}{\sqrt{x^2 + 1}}}{2 \left[ \frac{x^2 - (x^2 + 1)}{x^2 + 1} \right]} + c = \\
&= \frac{1}{4} \ln \left| \frac{x + \sqrt{x^2 + 1}}{x - \sqrt{x^2 + 1}} \cdot \frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 + 1}} \right| - \frac{x}{\sqrt{x^2 + 1}} \cdot \frac{x^2 + 1}{-2} + c = \\
&= \frac{1}{4} \ln \left| \frac{(x + \sqrt{x^2 + 1})^2}{x^2 - (x^2 + 1)} \right| + \frac{x}{2} \cdot \sqrt{x^2 + 1} + c = \frac{1}{4} \ln \frac{(x + \sqrt{x^2 + 1})^2}{|-1|} + \frac{x}{2} \cdot \sqrt{x^2 + 1} + c = \\
&= \underline{\underline{\frac{1}{2} \cdot \ln(x + \sqrt{x^2 + 1}) + \frac{x}{2} \cdot \sqrt{x^2 + 1} + c}}
\end{aligned}$$

**19. d)**  $[x \in \mathbb{R}]$   $\int \sqrt{x^2 + 1} \, dx = \left| \begin{array}{l} x = \frac{1}{2} \cdot (e^y - e^{-y}) \\ dx = \frac{1}{2} \cdot (e^y + e^{-y}) \, dy \end{array} \right| =$

$$\begin{aligned}
&= \int \sqrt{\left[ \frac{1}{2} \cdot (e^y - e^{-y}) \right]^2 + 1} \cdot \frac{1}{2} \cdot (e^y + e^{-y}) \, dy = \\
&= \int \sqrt{\frac{1}{4} \cdot (e^{2y} - 2 \underbrace{e^y e^{-y}}_{e^0=1} + e^{-2y}) + 4} \cdot \frac{1}{2} \cdot (e^y + e^{-y}) \, dy = \\
&= \int \sqrt{\frac{1}{4} \cdot (e^{2y} + 2 + e^{-2y})} \cdot \frac{1}{2} \cdot (e^y + e^{-y}) \, dy = \\
&= \int \sqrt{\left[ \frac{1}{2} \cdot (e^y + e^{-y}) \right]^2} \cdot \frac{1}{2} \cdot (e^y + e^{-y}) \, dy = \frac{1}{4} \int (e^y + e^{-y})^2 \, dy = \\
&= \frac{1}{4} \int (e^{2y} + 2 + e^{-2y}) \, dy = \frac{1}{4} \cdot \left( \frac{e^{2y}}{2} + 2y + \frac{e^{-2y}}{-2} \right) + c =
\end{aligned}$$

Jestliže:  $x = \frac{1}{2} \cdot (e^y - e^{-y})$  pak:  $y = \ln(x + \sqrt{x^2 + 1})$ , protože:



$$\begin{aligned}
x &= \frac{e^{\ln(x+\sqrt{x^2+1})} - e^{-\ln(x+\sqrt{x^2+1})}}{2} = \frac{1}{2} \cdot \left[ e^{\ln(x+\sqrt{x^2+1})} - e^{\ln(x+\sqrt{x^2+1})^{-1}} \right] = \\
&= \frac{1}{2} \cdot \left( x + \sqrt{x^2+1} - \frac{1}{x + \sqrt{x^2+1}} \right) = \\
&= \frac{1}{2} \cdot \left( x + \sqrt{x^2+1} - \frac{1}{x + \sqrt{x^2+1}} \cdot \frac{x - \sqrt{x^2+1}}{x - \sqrt{x^2+1}} \right) = \\
&= \frac{1}{2} \cdot \left[ x + \sqrt{x^2+1} - \frac{x - \sqrt{x^2+1}}{x^2 - (x^2+1)} \right] = \frac{1}{2} \cdot \left( x + \sqrt{x^2+1} - \frac{x - \sqrt{x^2+1}}{-1} \right) = x \\
&= \frac{1}{4} \cdot \left[ \frac{e^{2\ln(x+\sqrt{x^2+1})}}{2} + 2\ln(x + \sqrt{x^2+1}) + \frac{e^{-2\ln(x+\sqrt{x^2+1})}}{-2} \right] + c = \\
&= \frac{1}{8} \cdot (x + \sqrt{x^2+1})^2 + \frac{1}{2} \cdot \ln(x + \sqrt{x^2+1}) + \frac{1}{-8 \cdot (x + \sqrt{x^2+1})^2} + c = \\
&= \frac{1}{8} \cdot \left[ (x + \sqrt{x^2+1})^2 - \frac{1}{(x + \sqrt{x^2+1})^2} \cdot \frac{(x - \sqrt{x^2+1})^2}{(x - \sqrt{x^2+1})^2} \right] + \\
&\quad + \frac{1}{2} \cdot \ln(x + \sqrt{x^2+1}) + c = \\
&= \frac{1}{8} \cdot \left\{ (x + \sqrt{x^2+1})^2 - \frac{(x - \sqrt{x^2+1})^2}{[x^2 - (x^2+1)]^2} \right\} + \frac{1}{2} \cdot \ln(x + \sqrt{x^2+1}) + c = \\
&= \frac{1}{8} \cdot \left[ x^2 + 2x \cdot \sqrt{x^2+1} + (x^2+1) - \frac{x^2 - 2x \cdot \sqrt{x^2+1} + (x^2+1)}{(-1)^2} \right] + \\
&\quad + \frac{1}{2} \cdot \ln(x + \sqrt{x^2+1}) + c = \\
&= \frac{1}{8} \cdot 4x \cdot \sqrt{x^2+1} + \frac{1}{2} \cdot \ln(x + \sqrt{x^2+1}) + c = \\
&\quad = \underline{\underline{\frac{x}{2} \cdot \sqrt{x^2+1} + \frac{1}{2} \cdot \ln(x + \sqrt{x^2+1}) + c}}
\end{aligned}$$

$$\begin{array}{l|l}
 \text{19. e) } [x \in \mathbb{R}] & \int \sqrt{x^2 + 1} \, dx = \left| \begin{array}{l} \sqrt{x^2 + 1} = xw + 1 \\ x^2 + 1 = x^2w^2 + 2xw + 1 \\ 0 = x^2w^2 + 2xw - x^2 \\ 0 = x \cdot (x \cdot w^2 + 2w - x) \\ 0 = (x \cdot w^2 + 2w - x) \\ -2w = x(\cdot w^2 - 1) \\ x = \frac{-2w}{w^2 - 1} \\ dx = \frac{-2(w^2 - 1) - (-2w)2w}{(w^2 - 1)^2} \, dw \\ dx = \frac{2w^2 + 2}{(w^2 - 1)^2} \, dw \end{array} \right|
 \end{array}$$

$$= \int \left( \frac{-2w}{w^2 - 1} \cdot w + 1 \right) \cdot \frac{2w^2 + 2}{(w^2 - 1)^2} \, dw = \int \frac{-2w^2 + (w^2 - 1)}{w^2 - 1} \cdot \frac{2w^2 + 2}{(w^2 - 1)^2} \, dw =$$

$$= \int \frac{(-w^2 - 1) \cdot (2w^2 + 2)}{(w^2 - 1)^3} \, dw = \int \frac{-2w^4 - 2w^2 - 2w^2 - 2}{[(w + 1)(w - 1)]^3} \, dw =$$

$$= \int \left[ \frac{A}{w + 1} + \frac{B}{(w + 1)^2} + \frac{C}{(w + 1)^3} + \frac{D}{w - 1} + \frac{E}{(w - 1)^2} + \frac{F}{(w - 1)^3} \right] \, dw =$$

$$\left| \begin{array}{l} -2w^4 - 4w^2 - 2 = \\ = A(w+1)^2(w-1)^3 + B(w+1)(w-1)^3 + C(w-1)^3 + D(w+1)^3(w-1)^2 + E(w+1)^3(w-1) + F(w+1)^3 \\ w = 1: \quad -8 = 8F \quad \quad \quad F = -1 \\ w = -1: \quad -8 = -8C \quad \quad \quad C = 1 \\ w^5: \quad 0 = A + D \quad \quad -A = D \quad \quad (1) \\ w = 0: \quad -2 = -A - B - C + D - E + F \\ \quad \quad 0 = -B + 2D - E \quad \quad (2) \\ w = 2: \quad -50 = 9A + 3B + C + 27D + 27E + 27F \\ \quad \quad -24 = +3B + 18D + 27E \quad \quad (3) \\ w = -2: \quad -50 = -27A + 27B - 27C - 9D + 3E - F \\ \quad \quad -24 = +27B + 18D + 3E \quad \quad (4) \end{array} \right|$$

Rovnice (2); (3) a (4) řešíme Gaussovou eliminační metodou (sloupce B; D; E):

$$\left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 3 & 18 & 27 & -24 \\ 27 & 18 & 3 & -24 \end{array} \right] \begin{array}{c} (-3)(-27) \\ \downarrow \\ \downarrow \end{array} \sim \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 24 & 24 & -24 \\ 0 & 72 & -24 & -24 \end{array} \right] \begin{array}{c} (-3) \\ \downarrow \end{array} \sim \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 24 & 24 & -24 \\ 0 & 0 & -96 & 48 \end{array} \right]$$

a dostáváme:  $E = \frac{-1}{2}$ ;  $D = \frac{-1}{2}$ ;  $B = \frac{-1}{2}$ ;  $A = \frac{1}{2}$ . Tedy:

$$\begin{aligned}
&= \int \left[ \frac{\frac{1}{2}}{w+1} + \frac{\frac{-1}{2}}{(w+1)^2} + \frac{1}{(w+1)^3} + \frac{\frac{-1}{2}}{w-1} + \frac{\frac{-1}{2}}{(w-1)^2} + \frac{-1}{(w-1)^3} \right] dw = \\
&= \frac{1}{2} \int \frac{(w+1)'}{w+1} dw - \frac{1}{2} \int (w+1)^{-2} dw \quad | w+1 = u | \quad + \int (w+1)^{-3} dw - \\
&\quad - \frac{1}{2} \int \frac{(w-1)'}{w-1} dw - \frac{1}{2} \int (w-1)^{-2} dw \quad | w-1 = v | \quad - \int (w-1)^{-3} dw = \\
&= \dots = \underline{\underline{\frac{1}{2} \cdot \ln(x + \sqrt{x^2 + 1}) + \frac{x}{2} \cdot \sqrt{x^2 + 1} + c^3}}
\end{aligned}$$

20.  $[x \in (0; 1) \cup (1; \infty)]$

$$\begin{aligned}
&\int \frac{1}{\sqrt[3]{x} \cdot (1 - \sqrt{x})} dx = \int \frac{1}{x^{\frac{1}{3}} - x^{\frac{1}{3} + \frac{1}{2}}} dx = \\
&= \int \frac{1}{x^{\frac{1}{3}} - x^{\frac{5}{6}}} dx = \left| \begin{array}{l} \sqrt[6]{x} = y \quad (\text{viz př. 3}) \\ x = y^6 \\ dx = 6y^5 dy \\ x^{\frac{1}{6}} = y \end{array} \right| = \int \frac{1}{(y^6)^{\frac{1}{3}} - (y^6)^{\frac{5}{6}}} \cdot 6y^5 dy = \\
&= \int \frac{6y^5}{y^2 - y^5} dy = \int \frac{6y^2 \cdot y^3}{-y^2 \cdot (y^3 - 1)} dy = \int \frac{-6[(y^3 - 1) + 1]}{y^3 - 1} dy = \\
&= \int \frac{-6(y^3 - 1)}{y^3 - 1} dy + \int \frac{-6[1]}{(y - 1)(y^2 + y + 1)} dy = \\
&= -6 \int dy + \int \frac{A}{y - 1} dy + \int \frac{By + C}{y^2 + y + 1} dy = \\
&\quad \left| \begin{array}{l} -6 = A(y^2 + y + 1) + (By + C)(y - 1) \\ y = 1: -6 = 3A + 0 \quad A = -2 \\ y = 0: -6 = A - C \quad C = 4 \\ y^2: 0 = A + B \quad B = 2 \end{array} \right| \\
&= -6 \int dy + \int \frac{-2}{y - 1} dy + \int \frac{2y + 4}{y^2 + y + 1} dy =
\end{aligned}$$

<sup>3</sup> Tento výsledek (za použití funkce *logarc*) dává webová stránka  
[http://um.mendelu.cz/maw-html/index.php?lang=cs&form=integral&function=sqrt\(x^2%2B1\)](http://um.mendelu.cz/maw-html/index.php?lang=cs&form=integral&function=sqrt(x^2%2B1))

$$\begin{aligned}
&= -6y - 2 \int \frac{(y-1)'}{y-1} dy + \int \frac{(2y+1)+3}{y^2+y+1} dy = \\
&= -6y - 2 \ln|y-1| + \int \frac{(2y+1)}{y^2+y+1} dy + \int \frac{3}{y^2+y+\frac{1}{4}+\frac{3}{4}} dy = \\
&= -6y - \ln(y-1)^2 + \int \frac{(y^2+y+1)'}{y^2+y+1} dy + \int \frac{3}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}} dy = \\
&= \left| \begin{array}{l} y + \frac{1}{2} = \sqrt{\frac{3}{4}}u \\ dy = \frac{\sqrt{3}}{2} du \end{array} \right| = -6y - \ln(y-1)^2 + \ln(y^2+y+1) + \int \frac{3}{\frac{3}{4}u^2+\frac{3}{4}} \cdot \frac{\sqrt{3}}{2} du = \\
&= -6y + \ln \frac{y^2+y+1}{(y-1)^2} + \int \frac{3}{\frac{3}{4}(u^2+1)} \cdot \frac{\sqrt{3}}{2} du = \\
&= -6y + \ln \frac{y^2+y+1}{(y-1)^2} + \frac{4}{3} \cdot 3 \cdot \frac{\sqrt{3}}{2} \int \frac{1}{1+u^2} du = \\
&= -6y + \ln \frac{y^2+y+1}{(y-1)^2} + 2 \cdot \sqrt{3} \cdot \operatorname{arctg} u + c = \\
&= -6y + \ln \frac{y^2+y+1}{(y-1)^2} + 2 \cdot \sqrt{3} \cdot \operatorname{arctg} \frac{2y+1}{\sqrt{3}} + c = \\
&= -6 \cdot \sqrt[6]{x} + \ln \frac{\sqrt[6]{x^2} + \sqrt[6]{x} + 1}{(\sqrt[6]{x} - 1)^2} + 2 \cdot \sqrt{3} \cdot \operatorname{arctg} \frac{2 \cdot \sqrt[6]{x} + 1}{\sqrt{3}} + c
\end{aligned}$$


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**21.**  $[x \in (0; 1) \cup (1; \infty)]$   $\int \frac{1}{\sqrt{x} \cdot (1 - \sqrt[3]{x})} dx = \int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{2} + \frac{1}{3}}} dx =$

$$= \int \frac{1}{x^{\frac{1}{2}} - x^{\frac{5}{6}}} dx = \left| \begin{array}{l} \sqrt[6]{x} = z \quad (\text{viz př. 3}) \\ x = z^6 \\ dx = 6z^5 dz \\ x^{\frac{1}{6}} = z \end{array} \right| = \int \frac{1}{(z^6)^{\frac{1}{2}} - (z^6)^{\frac{5}{6}}} \cdot 6z^5 dz =$$

$$= \int \frac{6z^5}{z^3 - z^5} dz = \int \frac{6z^3 \cdot z^2}{-z^3 \cdot (z^2 - 1)} dz = \int \frac{-6 \cdot [(z^2 - 1) + 1]}{z^2 - 1} dz =$$

$$\begin{aligned}
&= \int \frac{-6(z^2 - 1)}{z^2 - 1} dz + \int \frac{-6[1]}{z^2 - 1} dz = -6 \int dz + \int \frac{-6}{(z+1)(z-1)} dz = \\
&= -6z + \int \frac{3}{z+1} dz - \int \frac{3}{z-1} dz = 3 \int \frac{(z+1)'}{z+1} dz - 3 \int \frac{(z-1)'}{z-1} dz - 6z = \\
&= 3 \ln |z+1| - 3 \ln |z-1| - 6x^{\frac{1}{6}} + c = 3 \ln \left| \frac{z+1}{z-1} \right| - 6 \cdot \sqrt[6]{x} = \\
&= \underline{\underline{3 \ln \frac{1 + \sqrt[6]{x}}{|-1 + \sqrt[6]{x}|} - 6 \cdot \sqrt[6]{x} + c}}
\end{aligned}$$

$$\begin{aligned}
\text{22. } [x \in (0; \infty)] \quad & \underline{\int \frac{x+3}{\sqrt{x}} dx} = \int \frac{x}{\sqrt{x}} dx + \int \frac{3}{\sqrt{x}} dx = \int x^{\frac{1}{2}} dx + 3 \int x^{-\frac{1}{2}} dx = \\
&= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 3 \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{\frac{2}{3} \cdot \sqrt{x^3} + 6 \cdot \sqrt{x} + c}}
\end{aligned}$$

$$\begin{aligned}
\text{23. } [x \in (0; \infty)] \quad & \underline{\int \frac{x^2+2}{\sqrt{x}} dx} = \int \frac{x^2}{\sqrt{x}} dx + \int \frac{2}{\sqrt{x}} dx = \int x^{\frac{3}{2}} dx + 2 \int x^{-\frac{1}{2}} dx = \\
&= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 2 \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{\frac{2}{5} \cdot \sqrt{x^5} + 4 \cdot \sqrt{x} + c}}
\end{aligned}$$

$$\begin{aligned}
\text{24. } [x \in (0; \infty)] \quad & \underline{\int \frac{x^2-1}{\sqrt{x}} dx} = \int \frac{x^2}{\sqrt{x}} dx - \int \frac{1}{\sqrt{x}} dx = \int x^{\frac{3}{2}} dx - \int x^{-\frac{1}{2}} dx = \\
&= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{\frac{2}{5} \cdot \sqrt{x^5} - 2 \cdot \sqrt{x} + c}}
\end{aligned}$$

$$\begin{aligned}
25. \quad [x \in \mathbb{R}] \quad \int \frac{1}{\sqrt{x^2 + 1}} \, dx &= \left| \begin{array}{l} \sqrt{x^2 + 1} = w - x \\ x^2 + 1 = w^2 - 2xw + x^2 \\ 2xw = w^2 - 1 \\ x = \frac{w^2 - 1}{2w} \\ dx = \frac{2w \cdot 2w - (w^2 - 1) \cdot 2}{(2w)^2} \, dw \\ dx = \frac{w^2 + 1}{2w^2} \, dw \end{array} \right| = \\
&= \int \frac{1}{w - \frac{w^2 - 1}{2w}} \cdot \frac{w^2 + 1}{2w^2} \, dw = \int \frac{1}{\frac{2w^2 - (w^2 - 1)}{2w}} \cdot \frac{w^2 + 1}{2w^2} \, dw = \\
&= \int \frac{2w}{2w^2 - w^2 + 1} \cdot \frac{w^2 + 1}{2w^2} \, dw = \int \frac{1}{w} \, dw = \ln |w| + c = \\
&= \ln \left| x + \underbrace{\sqrt{x^2 + 1}}_{\substack{>|x| \\ >0}} \right| + c = \underline{\underline{\ln(x + \sqrt{x^2 + 1}) + c}}
\end{aligned}$$