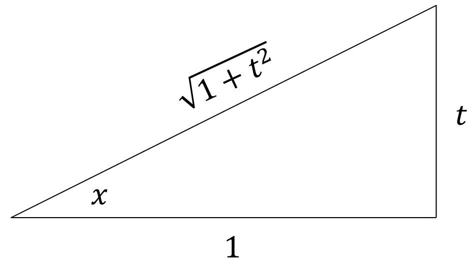
1. Najděte neurčitý integrál $\displaystyle\int \frac{1}{1+\sin^2 x}\,\mathrm{d}x \qquad D(f)=\mathbb{R}$

1.1. Substituce $\operatorname{tg} x = t$

V pravoúhlém trojúhelníku pro úhel x je: $\operatorname{tg} x = \dfrac{t}{1}$

Tím máme stanoveny velikosti obou odvěsen.
Velikost přepony určíme podle Pythagorovy věty.



$$\int \frac{1}{1+\sin^2 x}\,\mathrm{d}x = \left| \begin{array}{ll} \operatorname{tg} x = t & x = \operatorname{arctg} t \\ \sin x = \dfrac{t}{\sqrt{1+t^2}} & \mathrm{d}x = \dfrac{1}{1+t^2}\,\mathrm{d}t \end{array} \right|^{1} = \int \frac{1}{1+\frac{t^2}{1+t^2}} \cdot \frac{1}{1+t^2}\,\mathrm{d}t =$$

$$= \int \frac{1}{\frac{(1+t^2)+t^2}{1+t^2}} \cdot \frac{1}{1+t^2}\,\mathrm{d}t = \int \frac{1+t^2}{(1+t^2)+t^2} \cdot \frac{1}{1+t^2}\,\mathrm{d}t = \int \frac{1}{1+2\,t^2}\,\mathrm{d}t =$$

1.2. Substituce $t = \dfrac{s}{\sqrt{2}}$

$$= \int \frac{1}{1+2\,t^2}\,\mathrm{d}t = \left| \begin{array}{l} t = \dfrac{s}{\sqrt{2}} = \varphi(s) \\ s = \sqrt{2}\,t = \varphi^{-1}(s) \\ \mathrm{d}t = \dfrac{1}{\sqrt{2}}\,\mathrm{d}s \end{array} \right| = \int \frac{1}{1+2\cdot\left(\frac{s}{\sqrt{2}}\right)^2} \cdot \frac{1}{\sqrt{2}}\,\mathrm{d}s = \frac{1}{\sqrt{2}} \int \frac{1}{1+s^2}\,\mathrm{d}s =$$

$$= \frac{1}{\sqrt{2}} \operatorname{arctg} s = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\,t) = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\operatorname{tg} x) + \text{konst.}$$

Tedy $$\int \frac{1}{1+\sin^2 x}\,\mathrm{d}x = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\operatorname{tg} x) + \textbf{konst.}$$

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[1] Pokud $\operatorname{tg} x = t$ a požadujeme vyjádřit x (v závislosti na t), tak funkce $\operatorname{tg} x$ není prostá na celém svém definičním oboru. Proto k ní neexistuje inverzní funkce. Pro každý „kousek" je inverzní funkce jiná (viz pravý sloupec), ale derivace $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{1+t^2}$ je stejná pro libovolné x $(\forall x)$.

$x \in (\frac{-3\pi}{2}; \frac{-\pi}{2})$: $x = \operatorname{arctg} t - \pi$

$x \in (\frac{-\pi}{2}; \frac{\pi}{2})$: $x = \operatorname{arctg} t$

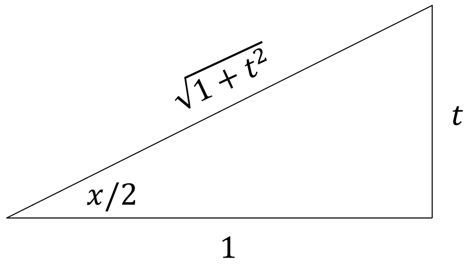
$x \in (\frac{\pi}{2}; \frac{3\pi}{2})$: $x = \operatorname{arctg} t + \pi$

atd.

2. Najděte neurčitý integrál $\displaystyle\int \sin^2 x \, \mathrm{d}x \qquad D(f) = \mathbb{R}$

Zde je několik způsobů řešení:

2.1. Substituce $\operatorname{tg}\frac{x}{2} = t$



V pravoúhlém trojúhelníku pro úhel $\frac{x}{2}$ je: $\quad \operatorname{tg}\frac{x}{2} = \frac{t}{1}$

Tím máme stanoveny velikosti obou odvěsen.
Velikost přepony určíme podle Pythagorovy věty.

$$\int \sin^2 x \, \mathrm{d}x = \left| \begin{array}{ll} \operatorname{tg}\frac{x}{2} = t & x = 2 \operatorname{arctg} t \\ \sin x = \dfrac{2\,t}{1+t^2} & \mathrm{d}x = \dfrac{2}{1+t^2}\,\mathrm{d}t \\ \sin 2\alpha = 2 \sin\alpha \cos\alpha & \sin x = \sin\left(2\frac{x}{2}\right) \end{array} \right| = \int \left(\frac{2\,t}{1+t^2}\right)^2 \cdot \frac{2}{1+t^2}\,\mathrm{d}t =$$

$$= \int \frac{8\,t^2}{(1+t^2)^3}\,\mathrm{d}t = \underbrace{\int \frac{8}{(1+t^2)^2}\,\mathrm{d}t}_{\mathcal{A}} + \int \frac{-8}{(1+t^2)^3}\,\mathrm{d}t = \operatorname{arctg} t + \frac{4\,t}{1+t^2} - \frac{3\,t^3 + 5\,t}{(1+t^2)^2} + K$$

Vrátit původní proměnnou!

2.1.1. $\mathcal{A} = \displaystyle\int \frac{8}{(1+t^2)^2}\,\mathrm{d}t = 4 \operatorname{arctg} t + \frac{4\,t}{1+t^2} + \textbf{konst.}$

$$\int \frac{8}{1+t^2}\,\mathrm{d}t = \left| \begin{array}{ll} u = \dfrac{1}{1+t^2} & v' = 8 \\ u' = \dfrac{-2\,t}{(1+t^2)^2} & v = 8\,t \end{array} \right| = \frac{8\,t}{1+t^2} - \int \frac{-16\,t^2}{(1+t^2)^2}\,\mathrm{d}t$$

Po rozkladu na parciální zlomky $\quad \Longrightarrow \quad \displaystyle \frac{8\,t}{1+t^2} - \int \left[\frac{16}{(1+t^2)^2} - \frac{16}{1+t^2}\right] \mathrm{d}t$

$$\int \frac{8}{1+t^2}\,\mathrm{d}t = \frac{8\,t}{1+t^2} - 2\int \frac{8}{(1+t^2)^2}\,\mathrm{d}t + \int \frac{16}{1+t^2}\,\mathrm{d}t$$

$$8\int \frac{1}{1+t^2}\,\mathrm{d}t = \frac{8\,t}{1+t^2} - 2\mathcal{A} + 16\int \frac{1}{1+t^2}\,\mathrm{d}t$$

$$2\mathcal{A} = \frac{8\,t}{1+t^2} + 8\int \frac{1}{1+t^2}\,\mathrm{d}t$$

$$\mathcal{A} = \dots$$

2.1.2. Analogicky

$$\int \frac{-8}{(1+t^2)^3}\,\mathrm{d}t = -3\,\mathrm{arctg}\,t - \frac{24\,t^3}{8\,t^4+16\,t^2+8} - \frac{4\,t}{8\,t^4+16\,t^2+8} + \textbf{konst.}$$

2.2. Substituce $\mathrm{tg}\,x = t$ (viz první příklad)

$$\int \sin^2 x\,\mathrm{d}x = \left| \begin{array}{ll} \mathrm{tg}\,x = t & x = \mathrm{arctg}\,t \\ \sin x = \dfrac{t}{\sqrt{1+t^2}} & \mathrm{d}x = \dfrac{1}{1+t^2}\,\mathrm{d}t \end{array} \right| = \int \frac{t^2}{1+t^2}\cdot\frac{1}{1+t^2}\,\mathrm{d}t = \int \frac{t^2}{(1+t^2)^2}\,\mathrm{d}t =$$

$$= \int \frac{(t^2+1)-1}{(1+t^2)^2}\,\mathrm{d}t = \int \frac{(t^2+1)}{(1+t^2)^2}\,\mathrm{d}t - \int \frac{1}{(1+t^2)^2}\,\mathrm{d}t = \int \frac{1}{1+t^2} - \left\{ \begin{array}{l} \text{výpočet integrálu } \mathcal{A} \\ \text{na předchozí straně} \end{array} \right\} =$$

$$= \frac{1}{2}\,\mathrm{arctg}\,t - \frac{t}{2(1+t^2)} + K = \frac{1}{2}\,\mathrm{arctg}\,(\,\mathrm{tg}\,x) - \frac{\mathrm{tg}\,x}{2(1+\,\mathrm{tg}^2 x)} + K$$

2.3. Per partes

$$\int \sin^2 x\,\mathrm{d}x = \left| \begin{array}{ll} u = \sin x & v' = \sin x \\ u' = \cos x & v = -\cos x \end{array} \right| = \sin x(-\cos x) - \int \cos x(-\cos x)\,\mathrm{d}x =$$

$$= -\sin x\cos x + \int \cos^2 x\,\mathrm{d}x = -\sin x\cos x + \int (1-\sin^2 x)\,\mathrm{d}x$$

$$\int \sin^2 x\,\mathrm{d}x = -\sin x\cos x + \int 1\,\mathrm{d}x - \int \sin^2 x\,\mathrm{d}x$$

$$2\int \sin^2 x\,\mathrm{d}x = -\sin x\cos x + x + \text{konst.}$$

$$\int \sin^2 x\,\mathrm{d}x = -\frac{1}{2}\sin x\cos x + \frac{1}{x} + K$$

2.4. Vzorec $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$$\int \sin^2 x\,\mathrm{d}x = \int \frac{1-\cos 2x}{2}\,\mathrm{d}x = \frac{1}{2}\int \mathrm{d}x - \frac{1}{2}\int \cos 2x dx = \frac{x}{2} - \frac{1}{2}\int \cos 2x dx =$$

$$= \left| \begin{array}{l} 2\,x = t \\ x = \dfrac{t}{2} \\ \mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}t \end{array} \right| = \frac{x}{2} - \frac{1}{2}\int \cos\left(2\frac{t}{2}\right)\frac{1}{2}\,\mathrm{d}t = \frac{x}{2} - \frac{1}{4}\int \cos t\,\mathrm{d}t = \frac{x}{2} - \frac{1}{4}\sin t + \text{konst.} =$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x + K$$

2.4.1. Odvození vzorce, když

$$\cos 2\,x = \cos^2 x - \sin^2 x$$
$$1 = \sin^2 x + \cos^2 x$$

$$\cos 2\,x = \cos^2 x - \sin^2 x$$
$$\cos 2\,x = (1 - \sin^2 x) - \sin^2 x$$
$$\cos 2\,x = 1 - 2\sin^2 x$$
$$2\sin^2 x = 1 - \cos 2\,x$$
$$\sin^2 x = \frac{1 - \cos 2\,x}{2}$$