APLIKACE DVOJNÉHO INTEGRÁLU

Př: Vypočtěte obsah rovinného obrazce:

1, $A: x^2+y=4, \quad x+y=2$

2, $A: y\le 1, \quad y\le x-1, \quad y\ge \ln x$

3, $A: xy=4, \quad x+y=5$

4, $A: x^2+y^2=4x, \quad x^2+y^2=2x, \quad y=x, \quad y=0$

Př: Vypočtěte objem tělesa:

1, $W: x+y+z=6, \quad 3x+2y=12, \quad y=0, \quad z=0$

2, $W: z=x^2+y^2, \quad x+y=1, \quad x=0, \quad y=0, \quad z=0$

Př: Vypočtěte obsah části plochy:

1, $S: 6x+3y+2z=12, \quad x\ge 0, \quad y\ge 0, \quad z\ge 0$

2, $S: 2z=x^2, \quad y\le 2x, \quad y\ge \frac{1}{2}x, \quad x\le 2\sqrt{2}$

Př: Určete těžiště tenké rovinné desky:

1, $A: x=1, \quad y=0, \quad y=\sqrt{x} \; ; \quad \sigma(x,y)=\frac{1}{x+1}$

APLIKACE DVOJNÉHO INTEGRÁLU

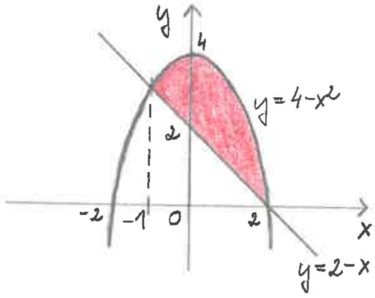
Př.: Vypočtěte obsah rovinného obrazce.

x libovolné
y libovolné

1) $A: \; x^2+y=4, \quad x+y=2$ ← přímka

parabola ↗ $y=4-x^2 \qquad y=2-x$

$$4-x^2=2-x$$
$$x^2-x-2=0$$
$$(x+1)(x-2)=0$$
$$x_1=-1,\; x_2=2$$



oblast I. druhu:

$$-1 \leq x \leq 2$$
$$2-x \leq y \leq 4-x^2$$

$$P=\iint_A dx\,dy=\int_{-1}^{2}\Big[\int_{2-x}^{4-x^2} dy\Big]dx=$$

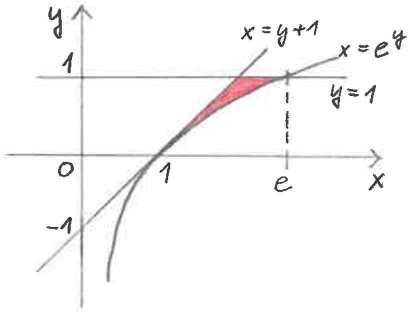
$$=\int_{-1}^{2}[y]_{2-x}^{4-x^2}dx=\int_{-1}^{2}(4-x^2-2+x)\,dx=\int_{-1}^{2}(2+x-x^2)\,dx=\Big[2x+\frac{x^2}{2}-\frac{x^3}{3}\Big]_{-1}^{2}=$$

$$=4+2-\frac{8}{3}+2-\frac{1}{2}-\frac{1}{3}=8-\frac{9}{3}-\frac{1}{2}=5-\frac{1}{2}=\underline{\frac{9}{2}}$$

2) $A: \; y\leq 1, \quad y\leq x-1, \quad y\geq \ln x$

$x=y+1 \qquad x=e^y$

Pozn: Přímka $y=x-1$ je tečna křivky $y=\ln x$ v bodě $[1,0]$.



oblast II. druhu:

$$0\leq y\leq 1$$
$$y+1\leq x\leq e^y$$

$$P=\iint_A dx\,dy=\int_0^1\Big[\int_{y+1}^{e^y}dx\Big]dy=\int_0^1[x]_{y+1}^{e^y}dy=\int_0^1(e^y-y-1)\,dy=$$

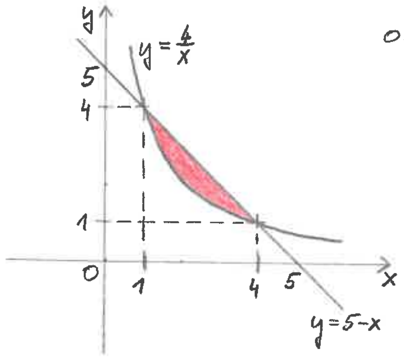
$$=\Big[e^y-\frac{y^2}{2}-y\Big]_0^1=e-\frac{1}{2}-1-\underbrace{1}_{e^0}=\underline{e-\frac{5}{2}}$$

3) $A:\ x\cdot y=4,\quad x+y=5$

hyperbola $\to y=\frac{4}{x}$ $\qquad y=5-x$

$$\frac{4}{x}=5-x \quad /\cdot x$$

$$x^2-5x+4=0$$
$$(x-1)(x-4)=0$$
$$x_1=1,\ x_2=4$$



oblast I. i II. druhu:

$1\le x\le 4$
$\frac{4}{x}\le y\le 5-x$

$$P=\iint_A dx\,dy=\int_1^4\Big[\int_{\frac{4}{x}}^{5-x}dy\Big]dx=$$

$$=\int_1^4[y]_{\frac{4}{x}}^{5-x}dx=\int_1^4\Big(5-x-\underbrace{\frac{4}{x}}_{4\cdot\frac{1}{x}}\Big)dx=\Big[5x-\frac{x^2}{2}-4\ln|x|\Big]_1^4=$$

$$=20-8-4\underbrace{\ln 4}_{\ln 2^2=2\ln 2}-5+\frac{1}{2}+4\underbrace{\ln 1}_{0}=\underline{\underline{\frac{15}{2}-8\ln 2}}$$

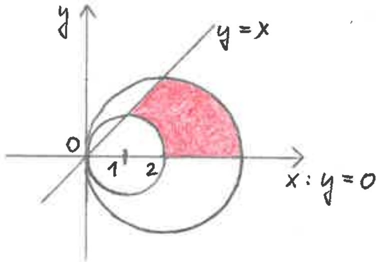
4) $A:\ x^2+y^2=4x,\quad x^2+y^2=2x,\quad y\le x,\quad y\ge 0$

$(x-2)^2+y^2=4$ $\qquad (x-1)^2+y^2=1$

$\to r^2=4r\cos\varphi$ $\qquad \to r^2=2r\cos\varphi$

$r=4\cos\varphi$ $\qquad r=2\cos\varphi$

$x = r.\cos\ \varphi$
$y = r.\sin\ \varphi$



Transformace do polárních souřadnic:

$0\le\varphi\le\frac{\pi}{4}$
$2\cos\varphi\le r\le 4\cos\varphi$

$$P=\iint_A dx\,dy=\int_0^{\frac{\pi}{4}}\Big[\int_{2\cos\varphi}^{4\cos\varphi}r\,dr\Big]d\varphi=$$

$$=\int_0^{\frac{\pi}{4}}\Big[\frac{r^2}{2}\Big]_{2\cos\varphi}^{4\cos\varphi}d\varphi=\frac{1}{2}\int_0^{\frac{\pi}{4}}(16\cos^2\varphi-4\cos^2\varphi)\,d\varphi=6\int_0^{\frac{\pi}{4}}\underbrace{\cos^2\varphi}_{\frac{1+\cos 2\varphi}{2}}\,d\varphi=$$

$$=3\int_0^{\frac{\pi}{4}}(1+\cos 2\varphi)\,d\varphi=3\Big[\varphi+\frac{1}{2}\sin 2\varphi\Big]_0^{\frac{\pi}{4}}=$$

$$=3\Big(\frac{\pi}{4}+\frac{1}{2}\underbrace{\sin\frac{\pi}{2}}_{1}-0-\frac{1}{2}\underbrace{\sin 0}_{0}\Big)=3\Big(\frac{\pi}{4}+\frac{1}{2}\Big)=\underline{\underline{\frac{3}{4}(\pi+2)}}$$

x libovolné
y libovolné

Př: Vypočtěte objem válcového tělesa.

1, W: $x+y+z=6$, $3x+2y=12$, $y=0$, $z=0$

$x+y+z=6$ → rovina

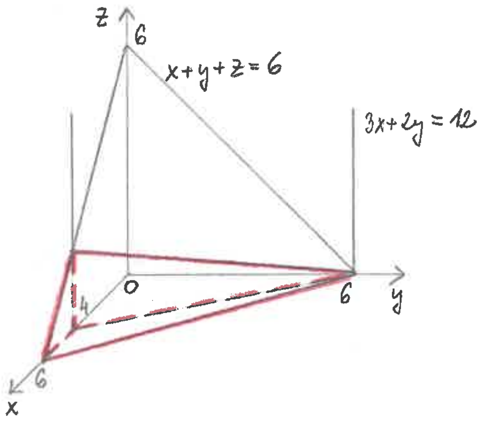
průsečnice s $z=0$: $\begin{cases} x+y=6 \end{cases}$

$x=6-y$

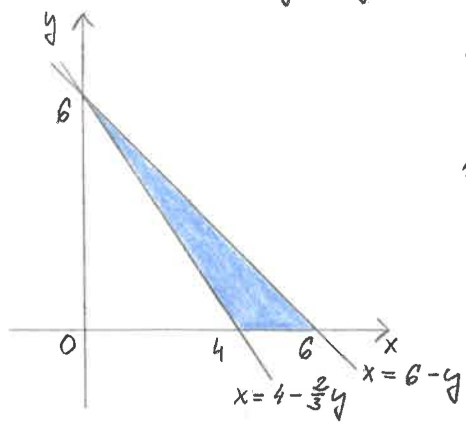
$3x+2y=12$: $x=4-\frac{2}{3}y$ → rovina kolmá k (x,y)

$f: z=6-x-y$

$g: z=0$



Průmět do roviny (x,y) → množina A:



oblast II. druhu

$$0 \le y \le 6$$
$$4-\tfrac{2}{3}y \le x \le 6-y$$

$$V=\iint_A (\overbrace{6-x-y}^{f(x,y)}-\overbrace{0}^{g(x,y)})\,dx\,dy=\int_0^6\Big[\int_{4-\frac{2}{3}y}^{6-y}(6-x-y)\,dx\Big]dy=\int_0^6\Big[6x-\frac{x^2}{2}-xy\Big]_{4-\frac{2}{3}y}^{6-y}dy=$$

$$=\int_0^6\Big(6(6-y)-\tfrac{1}{2}(6-y)^2-y(6-y)-6(4-\tfrac{2}{3}y)+\tfrac{1}{2}(4-\tfrac{2}{3}y)^2+y(4-\tfrac{2}{3}y)\Big)dy=$$

$$=\int_0^6\big(36-6y-18+6y-\tfrac{1}{2}y^2-6y+y^2-24+4y+8-\tfrac{8}{3}y+\tfrac{2}{9}y^2+4y-\tfrac{2}{3}y^2\big)dy=$$

$$=\int_0^6\big(2-\tfrac{2}{3}y+\tfrac{1}{18}y^2\big)dy=\Big[2y-\tfrac{1}{3}y^2+\tfrac{1}{54}y^3\Big]_0^6=12-\frac{36}{3}+\frac{6^3}{54}=$$

$$=12-12+\frac{\cancel{6}\cdot\cancel{6}^{\,2}\cdot\cancel{6}^{\,2}}{\cancel{6}\cdot\cancel{3}\cdot\cancel{3}}=\underline{\underline{4}}$$

Pozn: 1. Souřadnicové roviny:
(x,y) … $z=0$ … půdorysna
(x,z) … $y=0$ … nárysna
(y,z) … $x=0$ … bokorysna

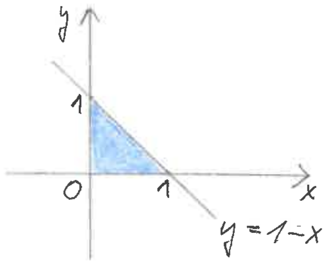
2. Těleso W je shora a zdola ohraničeno plochami $f(x,y)$ a $g(x,y)$ – v zadání příkladu je poznáme tak, že se v jejich rovnici vyskytuje proměnná z.

2) W: $\underbrace{z = x^2+y^2}_{f(x,y)}$, $x+y=1$, $x=0$, $y=0$, $\underbrace{z=0}_{g(x,y)}$

↳ rovina kolmá k (x,y)

rotační eliptický paraboloid s osou $o = z$

Průmět do roviny (x,y) → množina A



oblast I. i II. druhu

$0 \le x \le 1$
$0 \le y \le 1-x$

$$V = \iint_A (\overbrace{x^2+y^2}^{f(x,y)} - \overbrace{0}^{g(x,y)})\,dx\,dy = \int_0^1 \left[\int_0^{1-x} (x^2+y^2)\,dy\right] dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx =$$

$$= \int_0^1 \left(x^2(1-x) + \frac{1}{3}(1-x)^3\right) dx = \int_0^1 \left(x^2 - x^3 + \frac{1}{3} - x + x^2 - \frac{1}{3}x^3\right) dx =$$

$$= \int_0^1 \left(\frac{1}{3} - x + 2x^2 - \frac{4}{3}x^3\right) dx = \left[\frac{1}{3}x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4\right]_0^1 =$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \underline{\underline{\frac{1}{6}}}$$

x libovolné
y libovolné

Př: Vypočtěte obsah části plochy.

1) $\mathcal{S}: 6x+3y+2z=12, \quad x\geq 0, \quad y\geq 0, \quad z\geq 0$

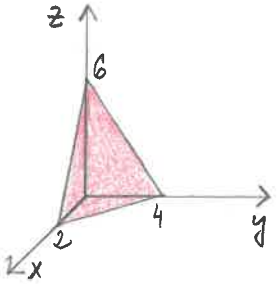
$\to \frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1$

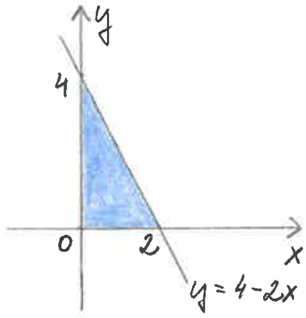
$\to f: z = 6-3x-\frac{3}{2}y$

$f'_x = -3, \quad f'_y = -\frac{3}{2}$

průsečnice s (x,y):
$z=0 \Rightarrow 6x+3y=12$
$y = 4-2x$

Průmět do $(x,y) \to$ množina A:





Oblast I. i II. druhu

$0 \leq x \leq 2$
$0 \leq y \leq 4-2x$

$$\mathcal{S} = \iint_A \sqrt{\underbrace{1+\underbrace{(-3)^2}_{(f'_x)^2}+\underbrace{\left(-\frac{3}{2}\right)^2}_{(f'_y)^2}}_{\sqrt{1+9+\frac{9}{4}}=\sqrt{\frac{49}{4}}=\frac{7}{2}}}\,dx\,dy = \iint_A \frac{7}{2}\,dx\,dy = \frac{7}{2}\int_0^2\left[\int_0^{4-2x} dy\right]dx = \frac{7}{2}\int_0^2 [y]_0^{4-2x}\,dx =$$

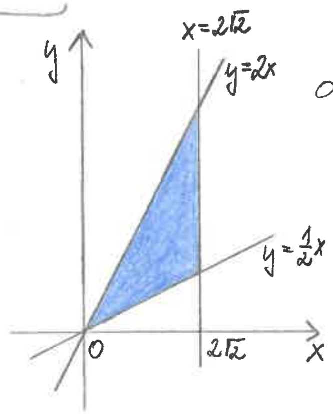
$$= \frac{7}{2}\int_0^2 (4-2x)\,dx = \frac{7}{2}\left[4x-x^2\right]_0^2 = \frac{7}{2}(8-4) = \frac{7}{2}\cdot 4 = \underline{14}$$

2) $\mathcal{S}: 2z = x^2, \quad \underbrace{y\leq 2x, \quad y \geq \frac{1}{2}x, \quad x \leq 2\sqrt{2}}_{A}$

$\to f: z = \frac{1}{2}x^2$

$f'_x = x, \quad f'_y = 0$

$\to$ Parabolická válcová plocha s řídící křivkou $2z=x^2$ ležící v rovině (x,z) a tvořícími přímkami rovnoběžnými s osou y.



A:

Oblast I. druhu

$0 \leq x \leq 2\sqrt{2}$
$\frac{1}{2}x \leq y \leq 2x$

$$\mathcal{S} = \iint_A \sqrt{1+x^2+0^2}\,dx\,dy = \int_0^{2\sqrt{2}}\left[\int_{\frac{1}{2}x}^{2x}\sqrt{1+x^2}\,dy\right]dx = \int_0^{2\sqrt{2}}\sqrt{1+x^2}\,[y]_{\frac{1}{2}x}^{2x}\,dx =$$

$$= \int_0^{2\sqrt{2}}\sqrt{1+x^2}\cdot\left(2x-\frac{1}{2}x\right)dx = \frac{3}{2}\int_0^{2\sqrt{2}} x\sqrt{1+x^2}\,dx = \left|\begin{array}{l} t = 1+x^2 \\ dt = 2x\,dx \end{array} \quad \begin{array}{c|c|c} x & 0 & 2\sqrt{2} \\ \hline t & 1 & 9 \end{array}\right| =$$

$$= \frac{3}{2}\cdot\frac{1}{2}\int_1^9 \overbrace{\sqrt{t}}^{=t^{\frac{1}{2}}}\,dt = \frac{3}{4}\left[\frac{\overbrace{t^{\frac{3}{2}}}^{=\sqrt{t^3}=t\sqrt{t}}}{\frac{3}{2}}\right]_1^9 = \frac{1}{2}\left[t\sqrt{t}\right]_1^9 = \frac{1}{2}(9\sqrt{9}-1\sqrt{1}) =$$

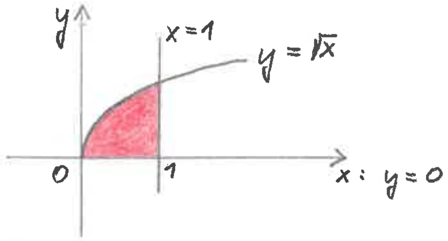
$$= \frac{1}{2}(27-1) = \underline{13}$$

Př: Určete těžiště tenké rovinné desky.

1) A: $x = 1,\ y \geq 0,\ y = \sqrt{x}\ ;\ \sigma(x,y) = \frac{1}{x+1}$

$x \neq -1$

y libovolné



Oblast I. i II. druhu

$0 \leq x \leq 1$

$0 \leq y \leq \sqrt{x}$

$$m = \iint_A \sigma(x,y)\,dx\,dy = \int_0^1 \Big[\int_0^{\sqrt{x}} \frac{1}{x+1}\,dy\Big]dx = \int_0^1 \frac{1}{x+1}\cdot[y]_0^{\sqrt{x}}\,dx = \int_0^1 \frac{\sqrt{x}}{x+1}\,dx =$$

$$= \left| \begin{array}{c} t = \sqrt{x} \\ t^2 = x \\ 2t\,dt = dx \end{array} \quad \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 0 & 1 \end{array} \right| = \int_0^1 \frac{t}{t^2+1}\cdot 2t\,dt = 2\int_0^1 \underbrace{\frac{t^2+1-1}{t^2+1}}_{\frac{t^2+1}{t^2+1} - \frac{1}{t^2+1} = 1 - \frac{1}{t^2+1}}\,dt =$$

$$= 2\int_0^1 \Big(1 - \frac{1}{t^2+1}\Big)dt = 2\,[t - \operatorname{arctg} t]_0^1 = 2(1 - \underbrace{\operatorname{arctg} 1}_{\frac{\pi}{4}} - 0 + \underbrace{\operatorname{arctg} 0}_{0}) =$$

$$= 2\Big(1 - \frac{\pi}{4}\Big) = \frac{1}{2}(4 - \pi)$$

$$S_y = \iint_A x\cdot\sigma(x,y)\,dx\,dy = \int_0^1 \Big[\int_0^{\sqrt{x}} \frac{x}{x+1}\,dy\Big]dx = \int_0^1 \frac{x}{x+1}\cdot[y]_0^{\sqrt{x}}\,dx = \int_0^1 \frac{x\sqrt{x}}{x+1}\,dx =$$

$$= \int_0^1 \frac{t^2\cdot t}{t^2+1}\cdot 2t\,dt = 2\int_0^1 \frac{t^4-1+1}{t^2+1}\,dt = 2\int_0^1 \Big(\frac{\overbrace{(t^2+1)(t^2-1)}^{t^4-1}}{t^2+1} + \frac{1}{t^2+1}\Big)dt =$$

$$= \int_0^1 \Big(t^2 - 1 + \frac{1}{t^2+1}\Big)dt = 2\Big[\frac{t^3}{3} - t + \operatorname{arctg} t\Big]_0^1 = 2\Big(\frac{1}{3} - 1 + \frac{\pi}{4}\Big) =$$

$$= 2\Big(\frac{\pi}{4} - \frac{2}{3}\Big) = \frac{1}{6}(3\pi - 8)$$

$$S_x = \iint_A y\cdot\sigma(x,y)\,dx\,dy = \int_0^1 \Big[\int_0^{\sqrt{x}} \frac{y}{x+1}\,dy\Big]dx = \int_0^1 \frac{1}{x+1}\cdot\Big[\frac{y^2}{2}\Big]_0^{\sqrt{x}}\,dx = \frac{1}{2}\int_0^1 \frac{x+1-1}{x+1}\,dx =$$

$$= \frac{1}{2}\int_0^1 \Big(1 - \frac{1}{x+1}\Big)dx = \frac{1}{2}\,[x - \ln|x+1|]_0^1 = \frac{1}{2}(1 - \ln 2)$$

$$x_T = \frac{S_y}{m} = \frac{\frac{1}{6}(3\pi - 8)}{\frac{1}{2}(4-\pi)} = \frac{3\pi - 8}{12 - 3\pi}$$

$$y_T = \frac{S_x}{m} = \frac{\frac{1}{2}(1 - \ln 2)}{\frac{1}{2}(4-\pi)} = \frac{1 - \ln 2}{4 - \pi}$$

$$T = \Big[\frac{3\pi - 8}{12 - 3\pi}\ ;\ \frac{1 - \ln 2}{4 - \pi}\Big]$$