Dynamické systémy a antiterorismus

Dynamic Systems and Counter–Terrorism

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Souhrn:

Rizika spojená s teroristickými útoky jsou z hlediska jejich objektivního posouzení poměrně těžko uchopitelným problémem. Mezi jinými obory lidské činnosti je to i matematika, která se v průběhu času snaží popsat teroristické hrozby různými metodami. Většinou jde o vytvoření modelu, ve kterém probíhají vybrané jevy a který dává interpretovatelný výstup. Jedná se o soustavy diferenciálních rovnic, Markovské procesy, teorii pravděpodobnosti, grafů či řízení, využívají se různé podoby statistických metod. V tomto článku se autoři pokusili o analýzu a srovnání některých vybraných metod s tím, že závěry jak metod samotných tak jejich hodnocení nejsou příliš spolehlivé, neboť výzkum zatím není dostatečně rozvinutý na to, aby uvažovaný model obsáhnul problém jako celek.

Summary:

The risks connected with terrorist attacks appear to be relatively difficult to solve from the point of view of objective evaluation. It is also mathematics which tries to describe terrorist threats by various methods over time. In most cases, a model is created in which the considered events take place and which makes an interpretable output possible. These can be systems of differential equations, Markov chains, theory of probability, graphs or operation etc. Various forms of statistical methods are used too. In this paper, the authors have tried to analyze and compare some selected methods. However, the conclusions of both the methods themselves and their evaluation are not very reliable because the current state of science is not developed enough to encompass the whole problem in one model.

Klíčová slova: Matematika, terorismus, riziko, modely, systémy, pravděpodobnost

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1. Introduction

During the last few decades there has been an effort to find a mathematical description of the fight against a growing enemy — terrorist organizations. Ways of doing this can vary. The mathematical means used for this goal are different but among the most frequent are dynamic systems. The aim of this paper is to show and comment some chosen ones. The method of some proves is included and the comparison of given approaches is made.

2. Lanchester equations and the method of their solution

The first military application of dynamic models comes from the early years of the previous century. It was Lanchester [Lanchester] who introduced the system of differential equations

$$\begin{aligned} \frac{\mathrm{d}A(t)}{\mathrm{d}t} &= -k_1 A(t)^{\alpha_1} D(t)^{\delta_1} \\ \frac{\mathrm{d}D(t)}{\mathrm{d}t} &= -k_2 A(t)^{\alpha_2} D(t)^{\delta_2} \end{aligned} \tag{1}$$

With the initial conditions $A(0) = A_0$, $D(0) = D_0$, where t represents time and must be greater than zero. $A(t)$ and $D(t)$ are functions depending on t. They represent – as the letters suggest – sizes of both armed forces — attackers and defenders. Effective destruction rates are described by coeficients k_1 and k_2, respectively, under conditions $k_1 > 0$ and $k_2 > 0$. All other parameters α_1, α_2, δ_1, δ_2 describe characteristics of the manner in which the battle is fought. For the classical war $\alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 1$ is proposed, for a more advanced way of war $\alpha_1 = 0$, $\alpha_2 = 1$, $\delta_1 = 1$, $\delta_2 = 0$ is proposed.

Both cases are solvable analytically.

Denote $A = A(t)$, $A' = \frac{\mathrm{d}A(t)}{\mathrm{d}t}$, $D = D(t)$, $D' = \frac{\mathrm{d}D(t)}{\mathrm{d}t}$ for simplification.

The $\alpha_1 = 0$, $\alpha_2 = 1$, $\delta_1 = 1$, $\delta_2 = 0$ case is easier to solve because the considered system $A' = -k_1 AD$, $D' = -k_2 AD$ becomes linear and can be solved by transformation to the second order equation:

Let's substitute $D = -\frac{1}{k_1} A'$ from the first equation into the second one: $-\frac{1}{k_1} A'' = -k_2 A$. This way we get $A'' - k_1 k_2 A = 0$. Then the characteristic equation is $\lambda^2 - k_1 k_2 = 0$ from where we get $\lambda_1 = \sqrt{k_1 k_2}$ and $\lambda_2 = -\sqrt{k_1 k_2}$. Finally

$$A = C_1 e^{\sqrt{k_1 k_2} \cdot t} + C_2 e^{-\sqrt{k_1 k_2} \cdot t} \tag{2}$$

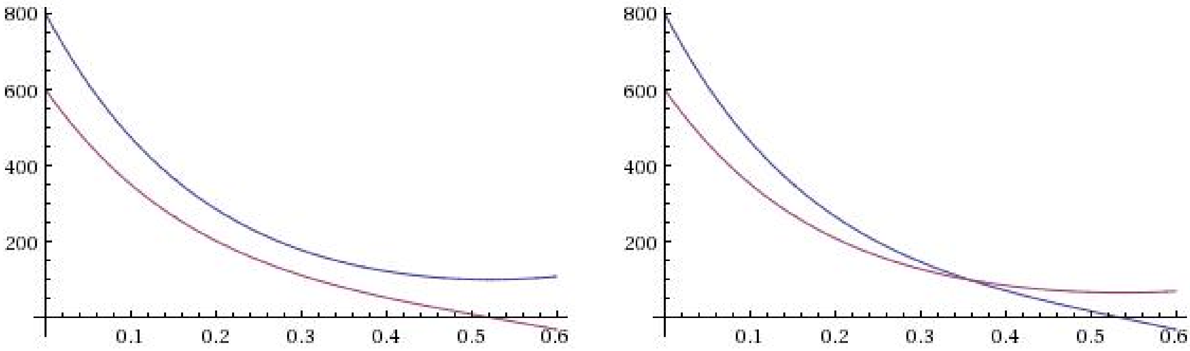
Now, substitute it to the first equation $C_1 \sqrt{k_1 k_2} \cdot e^{\sqrt{k_1 k_2} \cdot t} - C_2 \sqrt{k_1 k_2} \cdot e^{-\sqrt{k_1 k_2} \cdot t} = -k_1 D$ From there we get by integration

$$D = -\sqrt{\frac{k_2}{k-1}} \cdot \left[C_1 e^{\sqrt{k_1 k_2} \cdot t} - C_2 e^{-\sqrt{k_1 k_2} \cdot t} \right] \tag{3}$$

and our system is solved.

The discussion about it distinguishes three possibilities according to the value of the ratio $\frac{k_2}{k_1}$. If $\frac{k_2}{k_1} > [\frac{D_0}{A_0}]^2$ then the attackers win. If $\frac{k_2}{k_1} < [\frac{D_0}{A_0}]^2$ then the defenders win. We can see both cases in the following figures 1.

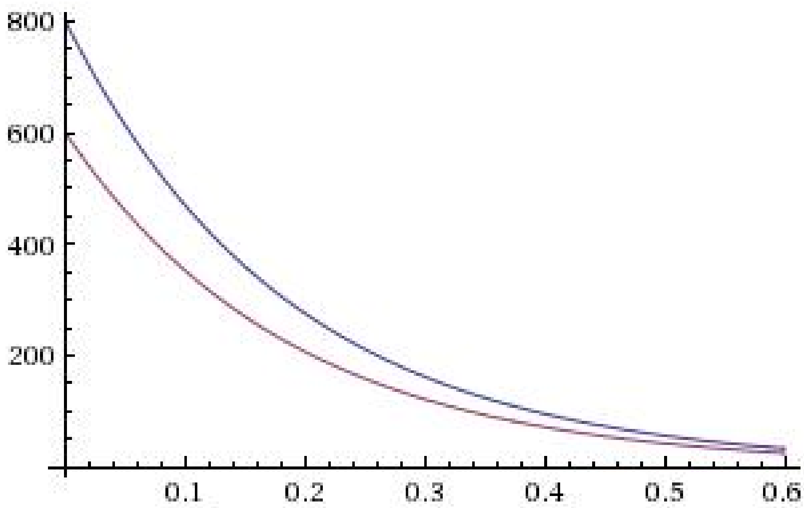
Figure 1:



Attackers win Defenders win

A special case occurs when $\frac{k_2}{k_1} = [\frac{D_0}{A_0}]^2$ because under this condition both armies will gradually sweep each other from the world. See the figure 2.

Figure 2:



Indecisive battle

The case $\alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 1$ is much more complicated but also interesting because its solution requires more steps. The system can be written as $A' = -k_1AD$, $D' = -k_2AD$. From the first equation we get

$$D = \frac{-1}{k_1} \cdot \frac{A'}{A} \tag{4}$$

and after substitution into the second one we obtain $-\frac{1}{k_1} \cdot \frac{A''.A - A'^2}{A^2} = -k_2A \cdot \frac{-A'}{k_1A}$. The equation can be easily modified to the shape

$$A'' - \frac{1}{A} \cdot A'^2 + k_2 \cdot A \cdot A' = 0 \tag{5}$$

Now, the $p(A) = A'(t)$ substitution is applied which will bring the equation to the linear form (denote $p(A)$ as p)

$$p' - \frac{1}{A} \cdot p = -k_2 \cdot A \tag{6}$$

After using the standard technique we come to the solution $p = A \cdot (c_1 - k_2 \cdot A)$.

This is a Bernoulli equation which is commonly solved by substitution $u = \frac{1}{A}$. When substituted and modified a little this equation becomes linear

$$u' - c_1 u + k_2 = 0 \tag{7}$$

Applying a common procedure we come to the solution

$$u = c_2 \cdot e^{c_1 \cdot t} - \frac{k_2}{c_1} \tag{8}$$

And after returning back from u to A we get the solution for attackers

$$A = \left[c_2 \cdot e^{c_1 \cdot t} - \frac{k_2}{c_1} \right]^{-1} \tag{9}$$

The solution for defenders D can be found by easy substitution A into (4). After some short simplifications we get:

$$D = \frac{c_1^2 \cdot c_2 \cdot e^{c_1 \cdot t}}{k_1 \cdot (c_1 \cdot c_2 e^{c_1 \cdot t} - k_2)} \tag{10}$$

Similarly to the previous case the discussion about the solution distinguishes three possibilities according to the value of the ratio $\frac{k_2}{k_1}$. The only difference is that unlike the quadratic relation shown a bit earlier now we have a linear relationship. So if $\frac{k_2}{k_1} > \frac{D_0}{A_0}$ then the attackers win. If $\frac{k_2}{k_1} < \frac{D_0}{A_0}$ then the defenders win. We can see both cases in the following figures. The third case $\frac{k_2}{k_1} = \frac{D_0}{A_0}$ is numerically not allowed, because it causes $c_1 = 0$ in the denominator.

3. Introducing a conditional probability

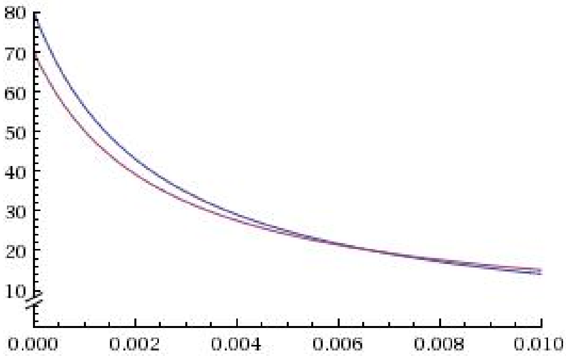
Lancester equations seem to be too (only) descriptive from the modern point of view and lacking some important properties like involvement of terrain or distinguishing between different weaponry. So different improvements have been derived to help the system to work better. One of them is the use of conditional probability, the use of which was introduced by [Perla, Lehoczky] and developed by [Gutfraind].

The basic equations from section 2 were replaced by

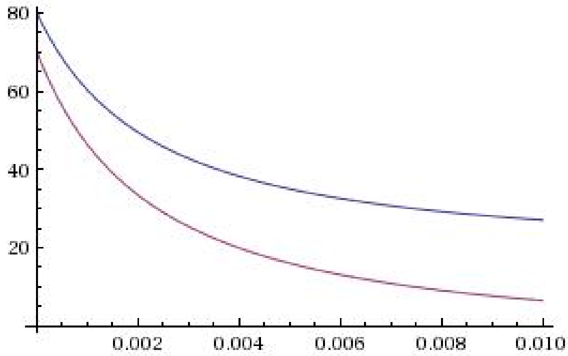
$$\begin{aligned} \mathrm{d}A &= -\tfrac{k_1}{\nu^q} \cdot A \cdot D \cdot \mathrm{d}t + \sigma_1 \cdot \mathrm{d}Z_1 \\ \mathrm{d}D &= -k_2 \cdot A \cdot \mathrm{d}t + \sigma_2 \cdot \mathrm{d}Z_2 \end{aligned} \tag{11}$$

The meaning of the newly introduced quantities is as follows: $\mathrm{d}Z_1$ and $\mathrm{d}Z_2$ are standard Brownian motions; σ_1 and σ_2 are appropriate standard deviations; ν represents

Figure 3:



Attackers win



Defenders win

a volume of the target; ν^q represents the transformation of the target dimension. This way the system takes into account stochastic approach as well as the influence of terrain and the asymmetric information about surprise attack. It is resistant to both weather and moral strength of soldiers too.

The solution procedure of the system can be found in [Powers]. By that the following expression is obtained. Denote $p = P(\text{target destruction}) = P(\text{attackers win})$ where p or $P(.)$ is a probability.

$$p = \frac{\Phi\left(\frac{\sqrt{2}}{\sigma} \cdot U(0)\right) - 0,5}{\Phi\left(\frac{\sqrt{2}}{\sigma}\right) - 0,5} \tag{12}$$

Where $U(A, D)$ is a function introduced during solution process to change the bivariate system to an univariate one and Φ is the standard normal distribution function.

Several interesting conclusions can be drawn if we compare $D(0)$ and $B = \sqrt{\frac{2\nu^q \cdot k_2 \cdot A(0)}{k_1}}$. If $D(0) \geq B$ then $p = 0$. It can be interpreted so that if it is possible to build a defensive force large enough the probability of attackers' victory is equal to zero.

There are more cases for $D(0) < B$:

$$\frac{\partial p}{\partial D(0)} < 0 \quad , \quad \lim_{D(0) \to \sqrt{\frac{2\nu^q \cdot k_2 \cdot A(0)}{k_1}}} p = 0 \quad , \quad \lim_{D(0) \to 0} p = 1 \tag{13}$$

The greater the initial defense forces are the lower the chance of target destruction is and vice versa. When $D(0)$ approaches 0 then target destruction becomes certain.

$$\frac{\partial p}{\partial A(0)} > 0 \quad , \quad \lim_{A(0) \to \infty} p = 1 \quad , \quad \lim_{A(0) \to \frac{k_1 \cdot [D(0)]^2}{2\nu^q \cdot k_2}} p = 0 \tag{14}$$

The greater the initial attack forces are the greater the chance of target destruction is and vice versa. When $A(0)$ approaches its lower limit the target destruction has no chance.

$$\frac{\partial p}{\partial \sigma} < 0 \quad , \quad \lim_{\sigma \to \infty} p > 0 \quad , \quad \lim_{\sigma \to 0} p = 1 \tag{15}$$

The greater the combat uncertainty is the lower the chance of target destruction is and vice versa. Total elimination is not possible. When σ approaches to zero then target destruction becomes certain.

$$\frac{\partial p}{\partial \nu^q} > 0 \quad , \qquad \lim_{\nu^q \to \infty} p < 1 \quad , \qquad \lim_{\nu^q \to \frac{k_1 \cdot [D(0)]^2}{2\nu^q \cdot k_2}} p = 0 \tag{16}$$

The greater the physical domain of the attack is the lower the chance of target destruction is. But complete certainty is impossible. When ν^q approaches its lower limit the target destruction cannot occur.

4. Strength of the organization

This approach addresses the problem of the number of leaders and foot soldiers changing with time. To formulate it the two variables are introduced. The letter L represents the number of leaders and the letter F represents the number of foot soldiers. Because the importance of a leader is more valuable than the importance of the foot soldier the strength of the organization is defined as a weighted sum [Gutfraind]

$$S = m \cdot L + F \quad , \qquad m > 1 \tag{17}$$

The approach assumes that both groups are weakened and refilled due to several reasons. It is proven that the growth rate of leaders is proportional to the number of foot soldiers with the parameter of proportionality p. Similarly the loss of a fraction of leaders per unit of time is modeled by the parameter d. Counter–terrorism measures also have significant influence in the removal of a number b of people per unit time. A constant rate of removal is preferred. These assumptions lead to the system of differential equations

$$\begin{aligned} L' &= p \cdot F - \mathrm{d}L - b \\ F' &= r \cdot (m \cdot L + F) - \mathrm{d}F - k \end{aligned} \tag{18}$$

The case of foot soldiers is modeled similarly. For the removal of a fraction d per unit time keeps (d equivalent to the one in the leaders' case for simplicity) and counter–terrorism measures cause the loss of k foot soldiers per unit time.

Variables $L = L(t)$, $F = F(t)$ as well as parameters p, d, b, r, m, and k under consideration depend on time and can be estimated by the least squares method.

Because this system is linear it can be easily solved by transformation to the 2^{nd} order non–homogenous linear differential equation

$$L'' + (d - r + d) \cdot L' + (d^2 - rd - rpm) \cdot L = rb - bd - pk \tag{19}$$

Considering initial conditions $F(0) = F_0$, $L(0) = L_0$ we have

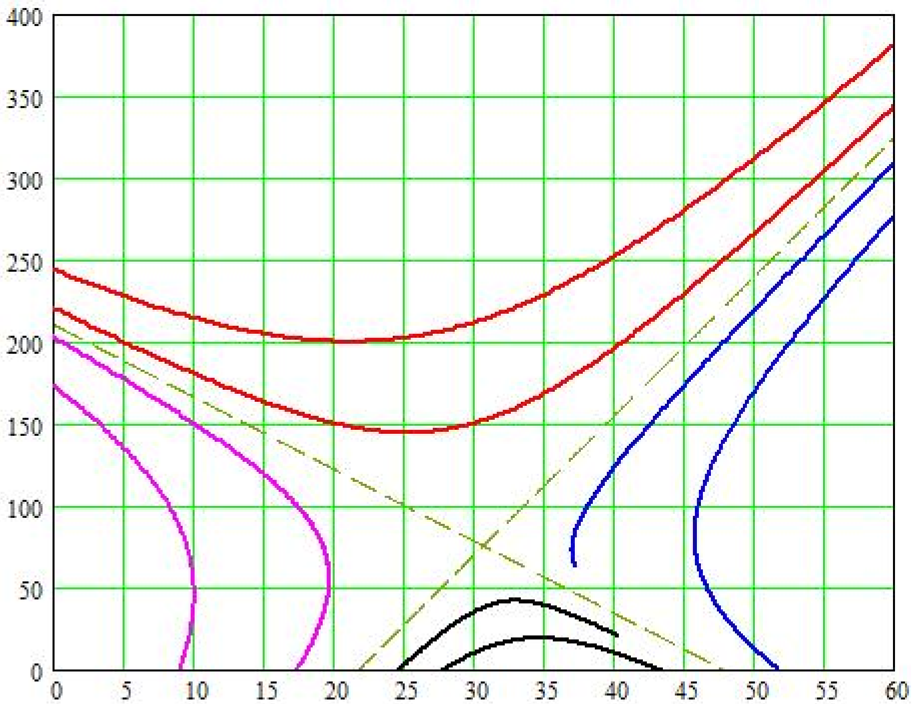
$$\begin{aligned} L(t) &= c_1 \cdot e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t} + \frac{rb - bd - pk}{d^2 - rd - rpm} \\ F(t) &= \frac{\lambda_1 + d}{p} \cdot c_1 \cdot e^{\lambda_1 t} + \frac{\lambda_2 + d}{p} \cdot c_2 \cdot e^{\lambda_2 t} + \frac{dpk + krmp}{rd + rpm - d^2} \end{aligned} \tag{20}$$

where

$$\begin{aligned} c_1 &= F_0 - \frac{\lambda_2 + d}{p} \cdot L_0 - \frac{(\lambda_2 + d) \cdot (rb - bd - pk) + p^2 \cdot (kd + brm)}{p \cdot (rd + rmp - d^2)} \\ c_2 &= L_0 - c_1 - \frac{rb - bd - pk}{d^2 - rd - rmp} \end{aligned} \tag{21}$$

When $L(t)$ is displayed on the horizontal axis and $F(t)$ on the vertical one for several different initial conditions we can see that the system of solutions has two asymptotes.

Figure 4:



$L(t) \times F(t)$ relation

They are showed by dashed lines. The one with negative slope can be called sink line which means that combinations of $L(t)$ and $F(t)$ below it represent cases when an organization is going to collapse. On the other hand combinations above it represent cases when organization remains working.

The other dash line can be called trend line which means that all surviving organizations will have approximately the same way of development in future independent of initial conditions.

5. Conclusion

All three examples of the dynamic system of the counter–terrorist battle are described by a system of two differential equations. But to compare them is not so easy for their philosophy differs.

The Lancaster equations represent a classical approach which has been discussed many times and which laid foundations for many followers. But even the best choice of coefficients α_1, α_2, δ_1, δ_2 with the aim to represent the most modern approach wouldn't bring the correct description of the actual counter–terrorist combat. But this theory was used in the World War II and represents the base for newer theories. The authors added their own solution of the system.

The introduction of probability has brought a more truthful picture of the problem and has included more aspects of real–world fight against terrorism. These solutions can show some interesting claims but they don't enable a lot of really new conclusions. In spite of this the derived relations represent a new step which can be further developed.

The effort to describe the lifecycle of a terrorist organization (attackers) by the changing number of leaders and foot soldiers seems to be of the most practical use. The solution shown in this paper uses the initial conditions at time $t = 0$ and fully corresponds to the results shown in [Gutfraind]. The division of organizations into categories according to their location vis–à–vis both the trend and sink lines is very instructive and gives clear results.

Of course the theory and practice can differ but the application of the shown approaches to the particular case is a different task. Then the theory has to be accompanied by practical testing to be reliable enough.

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